

TRANSLATION OF THE PAPER ‘MESURES  
STATIONNAIRES ET FERMÉS INVARIANTS DES  
ESPACES HOMOGÈNES I’, BY YVES BENOIST AND  
JEAN-FRANÇOIS QUINT, ANN. MATH. 174 (2011),  
TRANSLATED BY BARAK WEISS

ABSTRACT. A translation of the famous paper of Benoist and Quint.

The original abstract in English:

**Stationary measures and closed invariant subsets of homogeneous spaces.** Let  $G$  be a real simple Lie group,  $\Lambda$  be a lattice of  $G$  and  $\Gamma$  be a Zariski dense subsemigroup of  $G$ . We prove that every infinite  $\Gamma$ -invariant subset in the quotient  $X = G/\Lambda$  is dense. Let  $\mu$  be a probability measure on  $G$  whose support is compact and spans a Zariski dense subgroup of  $G$ . We prove that every atom free  $\mu$ -stationary probability measure on  $X$  is  $G$ -invariant. We also prove similar results for the torus  $X = \mathbb{T}^d$ .

## 1. INTRODUCTION

The goal of this text is to introduce a new technique in the study of stationary measures on homogeneous spaces, which we call the ‘exponential drift.’

**1.1. Motivation and principal results.** We will use it to prove:

**Theorem 1.1.** *Let  $G$  be a connected almost simple real Lie group,  $\Lambda$  a lattice in  $G$ ,  $X = G/\Lambda$  and  $\mu$  a probability measure on  $G$  with compact support, such that  $\text{supp } \mu$  generates a Zariski-dense subgroup of  $G$ . Then any non-atomic  $\mu$ -stationary Borel probability measure on  $X$  is the Haar measure on  $X$ .*

We now explain some of the (well-known) terminology used in the statement above. A real Lie group is *almost simple* if its Lie algebra is simple. A probability measure  $\nu$  on  $X$  is called  $\mu$ -stationary if  $\nu = \mu * \nu$ . It is called *non-atomic* if  $\nu(\{x\}) = 0$  for any  $x \in X$ . In case  $G$  is not a linear group, when we say that  $\Gamma$  is *Zariski dense* we mean that  $\text{Ad}(\Gamma)$  is Zariski dense in the linear group  $\text{Ad}(G)$  (where  $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$  is the adjoint representation). By *Haar measure* on  $X$  we mean the unique

$G$ -invariant probability measure on  $X$  induced by the Haar measure of  $G$ .

This theorem verifies a condition of *stiffness* of group actions introduced by Furstenberg [9] .

Ratner's theorems describe the measures on homogeneous spaces invariant and ergodic under a connected group generated by unipotents, as well as the orbit-closures. Shah and Margulis raised the question of extending these results to disconnected groups. We deduce an extension of Ratner's results for Zariski dense subgroups  $\Gamma$ , namely:

**Corollary 1.2.** *Let  $G, \Lambda, X, \Gamma$  be as in Theorem 1.1. Then:*

- a) *Any  $\Gamma$ -invariant non-atomic measure  $\nu$  is the Haar measure on  $X$ .*
- b) *Any closed  $\Gamma$ -invariant infinite set  $F \subset X$  is equal to  $X$ .*
- c) *Any sequence of distinct finite  $\Gamma$ -orbits  $X_n \subset X$  is equidistributed with respect to the Haar measure on  $X$ .*

A closed subset  $F \subset X$  is  $\Gamma$ -invariant if for any  $\gamma \in \Gamma$ ,  $\gamma F \subset F$ . Point c) means that the sequence of measures  $\nu_n := \frac{1}{\#X_n} \sum_{x \in X_n} \delta_x$  converges to the Haar measure on  $X$  with respect to the weak-\* topology. The simplest example in which one can apply the above results is for  $G = \mathrm{SL}_d(\mathbb{R})$ ,  $\Lambda = \mathrm{SL}_d(\mathbb{Z})$ ,  $d \geq 2$ , with  $\mu = \frac{1}{2}(\delta_{g_1} + \delta_{g_2})$  where the semigroup  $\Gamma$  generated by  $g_1, g_2$  is Zariski-dense. The space  $X$  is then the space of unimodular lattices in  $\mathbb{R}^d$ . Part c) generalizes previous results on equidistribution of Hecke orbits, obtained by Clozel-Oh-Ullmo.

Our method can be adapted to handle a larger class of homogeneous spaces. For instance, it makes it possible generalize a result of Bourgain, Furman, Lindenstrauss on Mozes as follows (in [2] the existence of proximal elements was assumed):

**Theorem 1.3.** *Let  $\Gamma$  be a sub-semigroup of  $\mathrm{SL}_d(\mathbb{Z})$  acting on  $\mathbb{R}^d$  strongly irreducibly. Let  $\mu$  be a measure on  $\mathrm{SL}_d(\mathbb{Z})$  whose finite support generates  $\Gamma$ . Then any non-atomic  $\mu$ -stationary probability measure on  $X = \mathbb{T}^d$  is the Haar measure of  $X$ .*

Recall that the action of  $\Gamma$  on  $\mathbb{R}^d$  is called *strongly irreducible* if any finite index subgroup of the group generated by  $\Gamma$ , acts irreducibly on  $\mathbb{R}^d$ . Note that in case a  $\mu$ -stationary measure  $\nu$  is atomic, it can be separated into a non-atomic and purely atomic part, and both measures in this decomposition are also  $\mu$ -stationary. Thus applying Theorem 1.1 or Theorem 1.3 we see that the non-atomic part is Haar. Regarding the purely atomic part of  $\nu$ , we will see (see Lemma 8.3) that it is a sum of a family of finitely supported  $\mu$ -stationary measures.

**Corollary 1.4.** *Let  $\Gamma$  be a subsemigroup of  $\mathrm{SL}_d(\mathbb{Z})$  acting strongly irreducibly on  $\mathbb{R}^d$ . Then:*

- a) *The only non-atomic  $\Gamma$ -invariant probability measure on  $X$  is the Haar measure.*
- b) *The only closed  $\Gamma$ -invariant infinite subset  $F \subset X$  is equal to  $X$ .*
- c) *Any sequence of distinct finite  $\Gamma$ -invariant sets  $X_n$  becomes equidistributed in  $X$  with respect to Haar measure.*

Assertion b) in Corollary 1.4 is due to Muchnik and to Guivarc'h-Starkov.

The approach of [2] is based on a delicate study of the Fourier coefficients of  $\nu$ . Our approach is purely ergodic-theoretic. For that reason it can be readily generalized to the case of homogeneous spaces. For example, Theorem 1.1 and Corollary 1.2 can be extended, with no significant change to the proof, to  $p$ -adic Lie groups  $G$ .

**1.2. Strategy.** Our approach is based on the study of the random walk on  $X = G/\Lambda$  (resp.  $X = \mathbb{T}^d$ ) induced by the random walk with law  $\mu$  on the group  $G$  (resp., on  $\mathrm{SL}_d(\mathbb{Z})$ ). In order to study the random walk we introduce a non-invertible dynamical system which we denote  $(B^{\tau,X}, \mathcal{B}^{\tau,X}, \beta^{\tau,X}, T_\ell^{\tau,X})$ . Without entering into too many details, we note that this dynamical system is fibered, with fiber  $X$ , over a suspension  $(B^\tau, \mathcal{B}^\tau, \beta^\tau, T^\tau)$  of a Bernoulli shift associated to  $\mu$ , and thus the space  $B^{\tau,X}$  is the product  $B^\tau \times X$ . The idea of using such a suspension was inspired by a paper of Lalley [13].

This dynamical system has two properties. Firstly, very simple formulas express the conditional expectation  $\phi_\ell := \mathbb{E}(\varphi | \mathcal{Q}_\ell^{\tau,X})$  of a bounded  $\mathcal{B}^{\tau,X}$ -measurable function  $\varphi$  on  $B^{\tau,X}$  relative to the  $\sigma$ -algebra  $\mathcal{Q}_\ell^{\tau,X} = (T_\ell^{\tau,X})^{-1} \mathcal{B}^{\tau,X}$  of events after a time  $\ell$ . Secondly, one has good control of the norm of products of elements of  $G$  associated with words appearing in these formulas of conditional expectation. In order to construct this dynamical system, one uses various classical theorems about random walks due in large part to Furstenberg: positivity of the first Lyapunov exponent, proximality of the walk induced on the flag variety, existence of limit probabilities  $\nu_b$  for the probabilities obtained as the image of the stationary measure  $\nu$  under a random word  $b$ .

Our main argument, which we call the *exponential drift*, is reminiscent of Ratner's idea which uses the Birkhoff ergodic theorem, replacing that theorem with Doob's Martingale convergence theorem. Its use was inspired by a paper of Bufetov [3]. This theorem allows us to assert

that the sequence  $\varphi_{c,\ell}$  converges, for  $\beta^{\tau,X}$ -a.e.  $(c, x)$  in  $B^{\tau,X}$ , to  $\varphi_\infty(c, x)$  where  $\varphi_\infty = \mathbb{E}(\varphi | \mathcal{Q}_\infty^{\tau,X})$  is the conditional expectation of  $\varphi$  with respect to the tail  $\sigma$ -algebra  $\mathcal{Q}_\infty^{\tau,X} = \bigcap_{\ell \geq 0} \mathcal{Q}_\ell^{\tau,X}$ . The idea is to compare  $\varphi_\ell(c, x)$  and  $\varphi_\ell(c, y)$  for two points  $x, y$  which are very close to each other and carefully chosen for the time  $\ell$ .

In order to start the drift argument, it is necessary to show that one may choose, when  $\nu$  is non-atomic, two points  $(c, x)$  and  $(c, y)$  which are not on the same stable leaf relative to the factor  $B^{\tau,X} \rightarrow B^\tau$ . This is a crucial point in our argument. It shows, roughly speaking, that the relative entropy of the fibered system is nonzero. In order to demonstrate this we exhibit a recurrence phenomenon for the random walk on  $X$ , analogous to the work of Eskin and Margulis [6], and combine this phenomenon with the ergodic theorem of Chacon-Ornstein.

In order to develop our exponential drift argument, it is necessary to obtain good control of norms of products of random matrices with law  $\mu$ , in the vector space  $V = \text{Lie}(G)$  (resp.  $V = \mathbb{R}^d$ ). The existence, due to Furstenberg, of an attracting limit subspace  $V_b$  is very useful.

When applying our drift argument, work remains. Unlike Ratner's argument, our argument only yields very patchy invariance properties for the stationary measures. For this reason we introduce a function which associates to each point  $(c, x)$ , a conditional measure  $\sigma(c, x)$  of the limit probability  $\nu_c$  along the foliation given by some limit subspace  $V_c$ . We identify all the spaces  $V_c$  thus constructed with the action of a unique vector space  $V_0$ , an action which we call the *horocyclic flow* and denote by  $\Phi_v$ . This point is important because it makes it possible to consider, as in [5], the function  $\sigma$  as a map taking values in a fixed vector space, the space of Radon measures on  $V_0$  up to normalization. It is this map  $\sigma$  to which we apply our drift argument. A crucial point is that the map  $\sigma$  is  $\mathcal{Q}_\infty^{\tau,X}$ -measurable. This results in commutation relations between  $\Phi_v$  and  $T_\ell^{\tau,X}$ , relations analogous to those existing in the hyperbolic plane between the geodesic and horocyclic flow.

The drift argument implies that the connected component  $J(c, x)$  of the stabilizer of  $\sigma(c, x)$  in  $V_0$  is almost surely nontrivial. This makes it possible to view the probability  $\nu_c$ , and hence  $\nu$ , as an average of probabilities  $\nu_{c,x}$  which are invariant under a nontrivial subspace  $J(c, x)$  of  $V_0$ .

In the case of the torus, one then deduces that the probabilities  $\nu_{c,x}$ , and hence  $\nu$ , are averages of probability measures supported on nontrivial subtori. Since the support of  $\mu$  acts strongly irreducibly on  $\mathbb{R}^d$ ,  $\nu$  is necessarily the Haar measure on  $\mathbb{T}^d$ .

In the case of a homogeneous space, an application of Ratner's theorems makes it possible to express  $\nu_{c,x}$  as an average of probability measures supported on orbits of nontrivial closed connected subgroups  $H$  of  $G$ . The  $G$ -invariance of  $\nu$  is deduced, thanks to a phenomenon of non-existence of  $\mu$ -stationary measures on the homogeneous space  $G/H$  with unimodular non-discrete stabilizer.

It is remarkable that our drift argument works even without it being necessary to explicitly describe the tail  $\sigma$ -algebra  $\mathcal{Q}_\infty^{\tau,X}$ . However, we will describe this tail  $\sigma$ -algebra in a forthcoming work and employ to this end the works of Blanchard, Conze, Guivarc'h, Raugi and Rohlin.

**1.3. Structure of the paper.** Chapters 2-5 collect the constructions and the properties of the dynamical systems associated with random walks that we will need.

Chapters 6-8 are devoted to the study of stationary measures on the spaces  $X = G/\Lambda$  and  $X = \mathbb{T}^d$ . These two cases will be treated simultaneously. We suggest to the reader to focus primarily on the case that  $X$  is the torus  $\mathbb{T}^2$ . Almost all of the arguments we shall develop are indispensable even for this case.

The goal of chapter 2 is formulas for the conditional expectation of fibrations and suspensions over non-invertible dynamical systems, including the remarkable 'law of the last jump.' Chapter 3 deals with some properties of stationary measures on Borel spaces equipped with a Borel action: existence of limit measures and the very useful phenomenon of recurrence off the diagonal. In chapter 4 we recall the construction of conditional measures along the orbits of a Borel action with discrete stabilizers. In chapter 5 we study linear strongly irreducible random walks. We recall the results of Furstenberg and introduce the dynamical system  $(B^\tau, \mathcal{B}^\tau, \beta^\tau, T^\tau)$  which is a suspension over a Bernoulli shift.

In chapter 6 we introduce the fibered dynamical system  $(B^{\tau,X}, \mathcal{B}^{\tau,X}, \beta^{\tau,X}, T^{\tau,X})$  associated to the random walk on  $X = G/\Lambda$  or  $X = \mathbb{T}^d$ . We check that this random walk satisfies not only the properties of recurrence off the diagonal which we will need in order to initiate the drift, but also the recurrence outside finite orbits which we will need in order to obtain topological consequences. We will also show non-existence of stationary measures on certain homogeneous spaces of semi-simple Lie groups, which we will require at the end of our study, for the space  $X = G/\Lambda$ . At the end of the chapter we will introduce the horocyclic flow  $\Phi_v$  on  $B^{\tau,X}$  and the conditional horocyclic map  $\sigma$ , and check that  $\sigma$  is  $\mathcal{Q}_\infty^{\tau,X}$ -measurable.

In chapter 7 we will present our general drift argument, and apply it to the map  $(c, x) \mapsto \sigma(c, x)$ . In section 8 we exploit the invariance properties of stationary measures, which follow from the drift argument, enabling us to conclude the proofs of Theorems 1.1 and 1.3. We then easily deduce Corollaries 1.2 and 1.4.

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## 2. SUSPENSIONS AND EXTENSIONS

*The goal of this chapter is to obtain formulas for the conditional expectations with respect to the tail  $\sigma$ -algebras in suspensions and fibrations over non-invertible dynamical systems (Proposition 2.3 and Lemma 2.5).*

**2.1. Cohomologous functions.** *The following lemma makes it possible to restrict our attention to suspensions with positive roof functions.*

**Lemma 2.1.** *Let  $(B, \mathcal{B}, \beta)$  be a Lebesgue probability space, equipped with an ergodic measure preserving transformation  $T$ . Let  $\theta : B \rightarrow \mathbb{R}$  be an integrable function (that is  $\int_B |\theta| d\beta < \infty$ ) with  $\int_B \theta d\beta > 0$ . Then there is a positive function  $\varphi$  which is almost surely finite, and a positive integrable function  $\tau$ , such that*

$$\theta - \varphi \circ T + \varphi = \tau.$$

*The function  $\tau$  can be chosen to be bounded below by a constant  $\varepsilon_0 > 0$ . The function  $\tau$  can be chosen to be bounded if  $\theta$  is bounded.*

In other words, the function  $\theta$  is cohomologous to  $\tau$  via  $\varphi$ .

*Proof.* For  $p \geq 1$ , denote  $\theta_p = \theta + \theta \circ T + \cdots + \theta \circ T^{p-1}$  and let

$$\psi = \inf_{p \geq 1} \theta_p, \quad \tau = \max(\psi, 0), \quad \varphi = -\min(\psi, 0).$$

By the Birkhoff ergodic theorem, for  $\beta$ -a.e.  $b$  in  $B$ ,  $\theta_p(b) \rightarrow_{p \rightarrow \infty} \infty$ . This implies that for almost all  $b$ , the inf in the definition of  $\psi(b)$  is a min and  $\varphi(b)$  is finite. Since  $\psi \leq \theta$ , we find  $\tau \leq \max(\theta, 0)$  and hence  $\tau$  is integrable. Finally, by definition,

$$\tau - \varphi = \psi = \min(\theta, \theta + \psi \circ T) = \theta + \min(0, \psi \circ T) = \theta - \varphi \circ T.$$

In order to obtain  $\tau$  which is bounded below by  $\varepsilon_0$ , apply the previous reasoning to the function  $\theta - \varepsilon_0$ . This is possible whenever  $\varepsilon_0 < \int_B \theta d\beta$ . The function  $\tau$  given in the construction is bounded when  $\theta$  is.  $\square$

**2.2. Suspension of a non-invertible system.** *We define in this section the suspension of a dynamical system where the roof function has a factor taking values in a compact group.*

Let  $(B, \mathcal{B}, \beta)$  be a Lebesgue probability space, equipped with an ergodic measure preserving transformation  $T$ . Let  $M$  be a compact metrizable topological group and

$$\tau = (\tau_{\mathbb{R}}, \tau_M) : B \rightarrow \mathbb{R} \times M$$

a measurable map such that  $\tau_{\mathbb{R}} : B \rightarrow \mathbb{R}$  is a positive integrable function. For any  $p \geq 0$ , and for  $\beta$ -a.e.  $b$  in  $B$ , denote

$$\tau_{\mathbb{R},p} = \tau_{\mathbb{R}}(T^{p-1}b) + \cdots + \tau_{\mathbb{R}}(b)$$

and

$$\tau_{M,p}(b) = \tau_M(T^{p-1}b) \cdots \tau_M(b).$$

Define the suspension  $(B^\tau, \mathcal{B}^\tau, \beta^\tau, T^\tau)$  as follows. The space  $B^\tau$  is

$$B^\tau = \{c = (b, k, m) \in B \times \mathbb{R} \times M : 0 \leq k < \tau_{\mathbb{R}}(b)\},$$

the measure  $\beta^\tau$  is obtained by normalizing the restriction to  $B^\tau$  of the product measure of  $\beta$  and the Haar measure of  $\mathbb{R} \times M$ , the  $\sigma$ -algebra  $\mathcal{B}^\tau$  is the product  $\sigma$ -algebra, and for almost every  $\ell \in \mathbb{R}_+$  and  $c = (b, k, m) \in B^\tau$ ,

$$T_\ell^\tau(c) = (T^{p_\ell(c)}b, k + \ell - \tau_{\mathbb{R},p_\ell(c)}(b), \tau_{M,p_\ell(c)}(m))$$

where

$$p_\ell(c) = \max\{p \in \mathbb{N} : k + \ell - \tau_{\mathbb{R},p}(b) \geq 0\}.$$

The flow  $T_\ell^\tau$  is then defined for all positive times.

**Lemma 2.2.** *The semigroup  $(T_\ell^\tau)$  of transformations of  $B^\tau$  preserves the measure  $\beta^\tau$ .*

*Proof.* The simplest approach is to avoid all calculations and consider  $(B^\tau, \mathcal{B}^\tau, \beta^\tau, T^\tau)$  as a factor of the suspension  $(\tilde{B}^\tau, \tilde{\mathcal{B}}^\tau, \tilde{\beta}^\tau, \tilde{T}^\tau)$  of the natural extension  $(\tilde{B}, \tilde{\mathcal{B}}, \tilde{\beta}, \tilde{T})$  of  $(B, \mathcal{B}, \beta, T)$  and reduce to the case when  $T$  is invertible.

When  $T$  is invertible, one can identify the suspended dynamical system as the quotient of the product  $B \times \mathbb{R} \times M$  by the transformation  $S : (b, k, m) \mapsto (Tb, k - \tau_{\mathbb{R}}(b), \tau_M(b)m)$ . The flow  $T_\ell^\tau$  is induced by the flow  $\tilde{T}_\ell^\tau$  preserving the product measure on  $B \times \mathbb{R} \times M$ . Therefore  $T_\ell^\tau$  preserves  $\beta^\tau$ .  $\square$

We remark finally, that it follows from the Birkhoff ergodic theorem, that for  $\beta^\tau$ -almost every  $c \in B^\tau$ ,

$$\lim_{p \rightarrow \infty} \frac{1}{p} \tau_{\mathbb{R},p}(b) = \int_B \tau_{\mathbb{R}} d\beta, \quad \lim_{\ell \rightarrow \infty} \frac{1}{\ell} p_\ell(c) = \frac{1}{\int_B \tau_{\mathbb{R}} d\beta}. \quad (2.1)$$

**2.3. The law of the last jump.** *We now establish the law of the last jump which plays a crucial role in controlling the drift, in §7.1. This law is an explicit formula for the conditional expectation of an event in  $B^\tau$  relative to  $(T_\ell^\tau)^{-1}(\mathcal{B}^\tau)$  when the base system is a Bernoulli shift.*

Let  $(A, \mathcal{A}, \alpha)$  be a Lebesgue probability space and  $(B, \mathcal{B}, \beta, T)$  a one-sided Bernoulli shift on the alphabet  $(A, \mathcal{A}, \alpha)$ , that is  $B = A^{\mathbb{N}}$ ,  $\beta = \alpha^{\otimes \mathbb{N}}$ ,  $\mathcal{B}$  is the product  $\sigma$ -algebra  $\mathcal{A}^{\otimes \mathbb{N}}$  and  $T$  is the right shift which sends  $b = (b_0, b_1, \dots) \in B$  to  $Tb = (b_1, b_2, \dots)$ . Let  $M$  be a metrizable compact topological group, let  $\tau = (\tau_{\mathbb{R}}, \tau_M) : B \rightarrow \mathbb{R} \times M$  be a measurable map such that  $\tau_{\mathbb{R}} : B \rightarrow \mathbb{R}$  is positive and integrable, and let  $(B^\tau, \mathcal{B}^\tau, \beta^\tau, T^\tau)$  be the suspension defined in §2.2.

We will require notation to parameterize the branches of the inverses of  $T_\ell^\tau$ . For  $q \geq 0$  and  $a, b \in B$ , we denote by  $a[q]$  the beginning of the word  $a$  written from right to left as  $a[q] = (a_{q-1}, \dots, a_1, a_0)$  and  $a[q]b \in B$  the concatenated word

$$a[q]b = (a_{q-1}, \dots, a_1, a_0, b_0, b_1, \dots, b_p, \dots).$$

For  $c = (b, k, m) \in B^\tau$  and  $\ell$  in  $\mathbb{R}_+$ , let  $q_{\ell,c} : B \rightarrow \mathbb{N}$  and  $h_{\ell,c} : B \rightarrow B^\tau$  the maps given, for  $a \in B$ , by

$$q_{\ell,c} = \tilde{q}_{\ell,c'} \quad \text{and} \quad h_{\ell,c} = \tilde{h}_{\ell,c'}, \quad \text{where} \quad c' = T_\ell^\tau(c)$$

and

$$\tilde{q}_{\ell,c}(a) = \min\{q \in \mathbb{N} : k - \ell + \tau_{\mathbb{R},q}(a[q]b) \geq 0\},$$

$$\tilde{h}_{\ell,c}(a) = (a[q]b, k - \ell + \tau_{\mathbb{R},q}(a[q]b), \tau_{M,q}(a[q]b)^{-1}m) \quad \text{with} \quad q = \tilde{q}_{\ell,c}(a).$$

By Birkhoff's theorem applied to the two-sided shift, for  $\beta$  a.e.  $a \in B$ , and  $\beta^\tau$  a.e.  $c \in B^\tau$ , one has the equality

$$\lim_{q \rightarrow \infty} \frac{1}{q} \tau_{\mathbb{R},q}(a[q]b) = \int_B \tau_{\mathbb{R}} d\beta > 0.$$

Hence the function  $\tilde{q}_{\ell,c}$  is almost surely finite and the image of the map  $\tilde{h}_{\ell,c}$  is the fiber  $(T_\ell^\tau)^{-1}(c)$ . The function  $q_{\ell,c}$  is thus also almost surely finite. In addition, for  $\beta$ -a.e.  $a \in B$ , for every  $q \geq 1$ , the function  $b \mapsto \tau_{\mathbb{R},q}(a[q]b)$  is  $\beta$ -integrable. Therefore by Birkhoff's ergodic theorem, for  $\beta^\tau$ -a.e.  $c \in B^\tau$ , one has

$$\lim_{p \rightarrow \infty} \frac{1}{p} \tau_{\mathbb{R},q}(a[q]T^p b) = 0$$



and hence, by (2.1), we have

$$\lim_{\ell \rightarrow \infty} q_{\ell,c}(a) = \infty. \quad (2.2)$$

Finally, the image of the map  $h_{\ell,c}$  is the fiber of  $T_\ell^\tau$  passing through  $c$ :

$$\{c'' \in B^\tau : T_\ell^\tau(c'') = T_\ell^\tau(c)\},$$

that is the atom of  $c$  in the partition associated with the  $\sigma$ -algebra  $(T_\ell^\tau)^{-1}(\mathcal{B}^\tau)$ .

**Proposition 2.3.** *The conditional expectation with respect to the  $\sigma$ -algebra  $(T_\ell^\tau)^{-1}(\mathcal{B}^\tau)$  is given, for any positive measurable function  $\varphi$  on  $B^\tau$  and for  $\beta^\tau$ -a.e.  $c = (b, k, m) \in B^\tau$ , by*

$$\mathbb{E}(\varphi | (T_\ell^\tau)^{-1}(\mathcal{B}^\tau))(c) = \int_B \varphi(h_{\ell,c}(a)) d\beta(a).$$

In other words, if we regard every element of the fiber of  $T_\ell^\tau$  over a point  $c' = (b', k', m') = T_\ell^\tau(c)$  in  $B^\tau$ , when completing the infinite word  $b'$  by the finite word  $a[q]$  written from right to left, the law of the finite word is obtained by randomly printing the letters  $a_i$ , independently with law  $\alpha$  in the alphabet  $A$ , where printing stops at time  $q_{\ell,c}(a)$ .

In particular, if  $\tau$  is bounded and if  $\ell \geq \sup \tau_{\mathbb{R}}$ , the law of the last jump  $a_0$  is  $\alpha$ . More generally, if  $\ell \geq q \sup \tau_{\mathbb{R}}$  the law of the last  $q$  jumps  $(a_{q-1}, \dots, a_0)$  is  $\alpha^{\otimes q}$ .

*Proof.* To simplify the notation used in the proof, we assume that  $M$  is trivial and thus  $\tau = \tau_{\mathbb{R}}$ . The general case of the proof is the same.

Introduce the function  $\varphi_0(c) = \int_B \varphi(\tilde{h}_{\ell,c}(a)) d\beta(a)$ . In order to show that the function  $\varphi_0 \circ T_\ell^\tau$  is the sought-after conditional expectation, it suffices to show that, for any positive  $\mathcal{B}^\tau$ -measurable function  $\psi$ , we have the equality

$$\int_{B^\tau} \psi(T_\ell^\tau c) \varphi(c) d\beta^\tau(c) = \int_{B^\tau} \psi(T_\ell^\tau c) \varphi_0(T_\ell^\tau c) d\beta^\tau(c). \quad (2.3)$$

To this end, we note that the left-hand side  $G$  is equal to

$$G = \sum_{p=0}^{\infty} \int_{B^\tau} \mathbf{1}_{\{p_\ell(c)=p\}} \psi(T^p b, k + \ell - \tau_p(n)) \varphi(b, k) d\beta(b) dk.$$

Introduce the variable  $c' = (b', k) = (T^p b, k + \ell - \tau_p(b)) \in B^\tau$  and  $a \in B$  such that  $a[p] = (b_0, \dots, b_{p-1})$ . One finds, when writing  $B(c', p) = \{a \in B : \tilde{q}_{\ell,c'}(a) = p\}$ , that

$$G = \int_{B^\tau} \psi(b', k') \sum_{p=0}^{\infty} \int_{B(c', p)} \varphi(a[p]b) d\beta(a) d\beta(b') dk',$$

and hence that

$$G = \int_{B^\tau} \psi(c') \int_B \varphi(\tilde{h}_{\ell, c'}(a)) d\beta(a) d\beta^\tau(c') = \int_{B^\tau} \psi(c') \varphi_0(c') d\beta^\tau(c').$$

Now (2.3) follows from the fact that  $T_\ell^\tau$  preserves the measure  $\beta^\tau$ .  $\square$

**2.4. Conditional expectation for the fibered system.** *We conclude this chapter with a general abstract lemma which constructs an invariant probability measure for the fibered dynamical system and by calculating its conditional expectation.*

Let  $(B, \mathcal{B})$  be a standard Borel space, i.e. isomorphic to a separable complete metric space with its Borel  $\sigma$ -algebra, and let  $\beta$  be a Borel probability measure on  $B$  and  $T$  an endomorphism of  $B$  preserving  $\beta$ . Let  $(X, \mathcal{X})$  be a standard Borel space,  $\pi : B \times X \rightarrow B$  the projection onto the first factor, and  $\hat{T}$  a measurable transformation of  $B \times X$  such that  $\pi \circ \hat{T} = T \circ \pi$ . Below we will write, for  $(b, x) \in B \times X$ ,

$$\hat{T}(b, x) = (Tb, \rho(b)x).$$

The space  $\mathcal{P}(X)$  of probability measures on  $(X, \mathcal{X})$  has itself the natural structure of a Borel space: this is the structure generated by the maps  $\mathcal{P}(X) \rightarrow \mathbb{R}$ ,  $\nu \mapsto \int_X \varphi d\nu$ , where  $\varphi : X \rightarrow \mathbb{R}$  is a bounded Borel function. If one realizes  $X$  as a compact metric space endowed with its Borel  $\sigma$ -algebra, this structure is generated by the maps  $\mathcal{P}(X) \rightarrow \mathbb{R}$ ,  $\nu \mapsto \int_X \varphi d\nu$  where  $\varphi : X \rightarrow \mathbb{R}$  is a continuous function. In particular, with respect to this Borel structure, the space  $\mathcal{P}(X)$  is also a standard Borel space.

Consider a  $\mathcal{B}$ -measurable collection  $B \rightarrow \mathcal{P}(X)$ ,  $b \mapsto \nu_b$  of probability measures on  $X$  such that for  $\beta$ -a.e.  $b \in B$ , we have

$$\nu_{Tb} = \rho(b)_* \nu_b. \quad (2.4)$$

We will denote by  $\lambda$  the Borel probability measure on  $(B \times X, \mathcal{B} \otimes \mathcal{X})$  defined by setting, for each positive Borel function  $\varphi : B \times X \rightarrow \mathbb{R}_+$ ,

$$\lambda(\varphi) = \int_B \int_X \varphi(b, x) d\nu_b(x) d\beta(b).$$

We will abbreviate this by writing

$$\lambda = \int_B \delta_b \otimes \nu_b d\beta(b). \quad (2.5)$$

**Lemma 2.4.** a) *The measure  $\lambda$  is  $\hat{T}$ -invariant and satisfies  $\pi_* \lambda = \beta$ .*

b) *Conversely, if  $T$  is invertible, then any  $\hat{T}$ -invariant probability measure on  $B \times X$  such that  $\pi_* \lambda = \beta$  is given by (2.5) for some measurable family of probabilities  $b \mapsto \nu_b$  satisfying (2.4).*

*Proof.* a) The  $\hat{T}$ -invariance of  $\lambda$  can be seen by a simple computation. For a  $(\mathcal{B} \otimes \mathcal{X})$ -measurable function  $\varphi : B \times X \rightarrow \mathbb{R}_+$ , one has

$$\begin{aligned} \int_{B \times X} \varphi(\hat{T}(b, x)) \lambda(b, x) &= \int_B \int_X \varphi(Tb, \rho(b)x) d\nu_b(x) d\beta(b) \\ &\stackrel{(2.4)}{=} \int_B \int_X \varphi(Tb, x) d\nu_{Tb}(x) d\beta(b) \\ &\stackrel{T_*\beta=\beta}{=} \int_B \int_X \varphi(b, x) d\nu_b(x) d\beta(b) \\ &= \int_{B \times X} \varphi(b, x) d\lambda(b, x). \end{aligned}$$

In case  $\varphi$  does not depend on the variable  $x$ , since the measures  $\nu_b$  are probabilities, one has

$$\begin{aligned} \int_{B \times X} \varphi(b, x) d\lambda(b, x) &= \int_B \int_X \varphi(b) d\nu_b(x) d\beta(b) \\ &= \int_B \varphi(b) d\beta(b). \end{aligned}$$

This implies  $\pi_*\lambda = \beta$ .

b) The probability measures  $\nu_b$  are the conditional probabilities of  $\lambda$  along the fibers of  $\pi$ . Since  $T$  is invertible, condition (2.4) follows from the  $\hat{T}$ -invariance of  $\lambda$  and uniqueness of conditional probabilities.  $\square$

We quickly recall the theorem of Rohlin [16] about disintegration of measures, which we will use below, and its relationship with conditional expectations.

Let  $\eta$  be a probability measure on a standard Borel space  $(Y, \mathcal{Y})$ . For any  $\sigma$ -algebra  $\mathcal{Y}' = p^{-1}(\mathcal{Z}) \subset \mathcal{Y}$  corresponding to a Borel factor  $p : (Y, \mathcal{Y}) \rightarrow (Z, \mathcal{Z})$ , we denote by  $y \mapsto \eta_y^{\mathcal{Y}'} \in \mathcal{P}(Y)$  the disintegration of  $\eta$  relative to  $\mathcal{Y}'$ . This is a  $\mathcal{Y}'$ -measurable map such that, for  $\eta$ -a.e.  $y \in Y$ ,  $\eta_y^{\mathcal{Y}'}$  is supported on  $p^{-1}(p(y))$  and one has

$$\eta = \int_Y \eta_y^{\mathcal{Y}'} d\eta(y). \quad (2.6)$$

This map  $y \mapsto \eta_y^{\mathcal{Y}'}$  is unique up to a set of  $\eta$ -measure zero.

In addition, for any  $\mathcal{Y}$ -measurable positive function  $\varphi : Y \rightarrow \mathbb{R}_+$ , for  $\eta$  a.e.  $y \in Y$ , one has

$$\mathbb{E}(\varphi | \mathcal{Y}')(y) = \int_B \varphi(y') d\eta_y^{\mathcal{Y}'}(y')$$

The following lemma asserts that the disintegration of  $\lambda$  with respect to the factor  $\hat{T} : B \times X \rightarrow B \times X$  can be easily derived from the disintegration of  $\beta$  with respect to the factor  $T : B \rightarrow B$ .

**Lemma 2.5.** *Assume that for  $\beta$ -a.e.  $b \in B$ , the map  $\rho(b) : X \rightarrow X$  is bijective. Then for every  $(\mathcal{B} \otimes \mathcal{X})$ -measurable and  $\lambda$ -integrable function  $\varphi : B \times X \rightarrow \mathbb{C}$  and for  $\lambda$ -a.e.  $(b, x) \in B \times X$ , we have*

$$\mathbb{E} \left( \varphi | \hat{T}^{-1}(\mathcal{B} \otimes \mathcal{X}) \right) (b, x) = \int_{B \times X} \varphi(b', \rho(b')^{-1} \rho(b)x) d\beta_b^{T^{-1}\mathcal{B}}(b'). \quad (2.7)$$

*Proof.* As explained above, for  $\lambda$ -a.e.  $(b, x) \in B \times X$ , one has the equality

$$\mathbb{E} \left( \varphi | \hat{T}^{-1}(\mathcal{B} \otimes \mathcal{X}) \right) (b, x) = \int_{B \times X} \varphi(b', x') d\lambda_{(b,x)}^{\hat{T}^{-1}(\mathcal{B} \otimes \mathcal{X})}(b', x').$$

Thus it remains to identify the measures  $\lambda_{(b,x)}^{\hat{T}^{-1}(\mathcal{B} \otimes \mathcal{X})}$ .

We note first that, since  $\rho(b)$  is bijective, for  $\lambda$ -a.e.  $(b, x) \in B \times X$ , the projection  $\pi$  induces a bijection of the fiber  $\hat{T}^{-1}(\hat{T}(b, x))$  with  $T^{-1}(Tb)$  where the inverse is given by  $b' \mapsto (b', \rho(b')^{-1} \rho(b)x)$ . Denote by  $\mu_{(b,x)}$  the measure on  $B \times X$  given by the right hand side in the sought-for equality (2.7):

$$\int_{B \times X} \varphi(b', x') d\mu_{(b,x)}(b', x') = \int_B \varphi(b', \rho(b')^{-1} \rho(b)x) d\beta_b^{T^{-1}\mathcal{B}}(b').$$

We want to show that for  $\lambda$ -a.e.  $(b, x) \in B \times X$ , we have

$$\lambda_{(b,x)}^{\hat{T}^{-1}(\mathcal{B} \otimes \mathcal{X})} = \mu_{(b,x)}.$$

To this end, first note that the map  $(b, x) \mapsto \mu_{(b,x)}$  is  $\hat{T}^{-1}(\mathcal{B} \otimes \mathcal{X})$ -measurable and that the measure  $\mu_{(b,x)}$  is supported on  $\hat{T}^{-1}(\hat{T}(b, x))$ . Secondly we will compute the following integral  $I$  for every  $\lambda$ -integrable function  $\varphi : B \times X \rightarrow \mathbb{C}$ :

$$I = \int_{B \times X} \int_{B \times X} \varphi(b', x') d\mu_{(b,x)}(b', x') d\lambda(b, x).$$

For  $\beta$ -a.e.  $b$  we apply Fubini's theorem in the space  $(B \times X, \mathcal{B} \otimes \mathcal{X}, \beta_b^{T^{-1}\mathcal{B}} \otimes \nu_b)$ , and obtain

$$I = \int_B \int_B \int_X \varphi(b', \rho(b')^{-1} \rho(b)x) d\nu_b(x) d\beta_b^{T^{-1}\mathcal{B}}(b') d\beta(b).$$

Using (2.4) one finds

$$\begin{aligned} I &= \int_B \int_B \int_X \varphi(b', \rho(b')^{-1}x) d\nu_{Tb}(x) d\beta_b^{T^{-1}\mathcal{B}}(b') d\beta(b) \\ &= \int_B \int_B \int_X \varphi(b', \rho(b')^{-1}x) d\nu_{Tb'}(x) d\beta_b^{T^{-1}\mathcal{B}}(b') d\beta(b). \end{aligned}$$

Finally, applying once more (2.4) and (2.6), one obtains

$$\begin{aligned} I &= \int_B \int_B \int_X \varphi(b', x) d\nu_{b'}(x) d\beta_b^{T^{-1}\mathcal{B}}(b') d\beta(b) \\ &= \int_B \int_X \varphi(b, x) d\nu_b(x) d\beta(b) = \int_{B \times X} \varphi(b, x) d\lambda(b, x). \end{aligned}$$

By uniqueness of the disintegration, we have the equality  $\lambda_{(b,x)}^{\hat{T}^{-1}(\mathcal{B} \otimes \mathcal{X})} = \mu_{(b,x)}$ , for  $\lambda$ -a.e.  $(b, x) \in B \times X$ .  $\square$

### 3. RANDOM WALKS ON $G$ -SPACES

*In this chapter we collect some fundamental properties of stationary measures which are valid in a very general context.*

**3.1. Stationary measures and Furstenberg measure.** *To each stationary probability measure  $\nu$  we associate a probabilistic dynamical system  $(B^X, \mathcal{B}^X, \beta^X, T^X)$ .*

Let  $G$  be a metrizable locally compact group,  $\mathcal{G}$  its Borel  $\sigma$ -algebra,  $\mu$  a Borel probability measure on  $G$  and  $(B, \mathcal{B}, \beta, T)$  the one-sided Bernoulli shift on the alphabet  $(G, \mathcal{G}, \mu)$ .

Let  $(X, \mathcal{X})$  be a standard Borel space equipped with a Borel action of  $G$ . Let  $\nu$  be a Borel probability measure on  $X$  which is  $\mu$ -stationary, i.e.  $\mu * \nu = \nu$ .

We denote by  $T^X$  the transformation on  $B^X = B \times X$  given by, for  $(b, x) \in B^X$ ,

$$T^X(b, x) = (Tb, b_0^{-1}x). \quad (3.1)$$

We denote, for  $n \geq 0$ , by  $\mathcal{B}_n$  the sub- $\sigma$ -algebra of  $\mathcal{B}$  generated by the coordinate functions  $b_i$ ,  $i = 0, 1, \dots, n$ , and denote by  $\pi : B^X \rightarrow B$  the projection onto the first factor.

**Lemma 3.1.** *Let  $\nu$  be a  $\mu$ -stationary probability measure on  $X$ .*

- a) *There is a unique probability measure  $\beta^X$  on  $(B^X, \mathcal{B} \otimes \mathcal{X})$  such that, for any  $n \geq 0$  and any  $\mathcal{B}_n \otimes \mathcal{X}$ -measurable bounded function  $\varphi$ ,*

$$\int_{B^X} \varphi(b, x) d\beta^X(b, x) = \int_{B^X} \varphi(b, b_0 \cdots b_{n-1}y) d\beta(b) d\nu(y). \quad (3.2)$$

- b) *The probability measure  $\beta^X$  is  $T^X$ -invariant and satisfies  $\pi_*\beta^X = \beta$ .*

*Proof.* a). For  $n \geq 0$  we introduce the probability measure on  $\mathcal{B}_n \otimes X$  defined by  $\beta_n^X = \int_B \delta_b \otimes (b_0 \dots b_{n-1})_* \nu d\beta(b)$ . Since  $\nu$  is  $\mu$ -stationary, for every  $n \geq 0$ , the measure  $\beta_{n+1}^X$  coincides with  $\beta_n^X$  on the  $\sigma$ -algebra  $\mathcal{B}_n \otimes \mathcal{X}$ . By the theorem of Caratheodory, it follows that there is a unique probability measure  $\beta^X$  on  $\mathcal{B} \otimes \mathcal{X}$  which coincides with  $\beta_n^X$  on  $\mathcal{B}_n \otimes \mathcal{X}$  for every  $n \geq 0$ .

b). For any  $n \geq 0$ , one has  $(T^X)^{-1}(\mathcal{B}_n \otimes \mathcal{X}) \subset (\mathcal{B}_{n+1} \otimes \mathcal{X})$  and, for any bounded  $\mathcal{B}_n \otimes \mathcal{X}$ -measurable function  $\varphi$ , by definition,

$$\begin{aligned} \int_{B^X} \varphi(T^X(b, x)) d\beta_{n+1}^X(b, x) &= \int_{B^X} \varphi(Tb, b_0^{-1}b_0b_1 \dots b_n y) d\beta(b) d\nu(y) \\ &= \int_{B^X} \varphi(b, x) d\beta_n^X(b, x). \end{aligned}$$

It follows that  $T_*^X \beta^X = \beta^X$ . In addition, equation (3.2) with  $n = 0$  gives the equality  $\pi_* \beta^X = \beta$ .  $\square$

We denote by  $\mathcal{B}^X$  the completion of the  $\sigma$ -algebra  $\mathcal{B} \otimes \mathcal{X}$  with respect to the measure  $\beta^X$ .

**3.2. Martingales and conditional probabilities.** *In this section, we associate with each stationary probability measure  $\nu$  on  $X$  a measurable and  $T$ -equivariant family  $(\nu_b)_{b \in B}$  of probability measures on  $X$ .*

The disintegration of  $\beta^X$  along the factor map  $\pi$ , proves the existence of a  $\mathcal{B}$ -measurable map  $B \rightarrow \mathcal{P}(X)$ ,  $b \mapsto \nu_b$ , such that

$$\beta^X = \int_B \delta_b \otimes \nu_b d\beta(b). \quad (3.3)$$

In other words, for any bounded  $\mathcal{B}^X$ -measurable function  $\varphi$  on  $B^X$ , one has

$$\beta^X(\varphi) = \int_B \int_X \varphi(b, y) d\nu_b(y) d\beta(b). \quad (3.4)$$

Also one has the following equality for  $\beta^X$ -a.e.  $(b, x) \in B^X$

$$\mathbb{E}(\varphi | \pi^{-1}\mathcal{B})(b, x) = \int_X \varphi(b, y) d\nu_b(y), \quad (3.5)$$

where the conditional expectation is taken relative to the probability measure  $\beta^X$ .

The following lemma interprets the conditional probabilities  $\nu_b$  as limit probabilities.

**Lemma 3.2.** *Let  $\nu$  be a  $\mu$ -stationary probability measure on  $X$  and let  $b \mapsto \nu_b$  be the  $\mathcal{B}$ -measurable family of probability measures on  $X$  constructed above.*

- a) *For any bounded Borel function  $f$  on  $X$ , for  $\beta$ -a.e.  $b \in B$ , we have*

$$\nu_b(f) = \lim_{p \rightarrow \infty} (b_{0*} \cdots b_{p*} \nu)(f). \quad (3.6)$$

- b) *For  $\beta$ -a.e.  $b \in B$ , we have*

$$\nu_b = b_{0*} \nu_{Tb}. \quad (3.7)$$

- c) *We have*

$$\nu = \int_B \nu_b d\beta(b). \quad (3.8)$$

- d) *The map  $b \mapsto \nu_b$  is the unique  $\mathcal{B}$ -measurable map  $B \rightarrow \mathcal{P}(X)$  for which (3.7) and (3.8) hold.*  
e) *Conversely, for any  $\mathcal{B}$ -measurable family  $b \mapsto \nu_b \in \mathcal{P}(X)$  satisfying (3.7), the measure  $\nu$  given by (3.8) is  $\mu$ -stationary.*

*Proof.* a). For  $\beta$ -a.e.  $b \in B$ , we denote by  $\nu_{b,p}$  the probability measure  $\nu_{b,p} = b_{0*} \cdots b_{p*} \nu \in \mathcal{P}(X)$ . The proof is based on an explicit formula for the conditional expectation: for each  $p \geq 0$ , for any bounded  $\mathcal{X}$ -measurable function  $f$ , which we will consider as a function on  $B^X$ , for  $\beta^X$ -a.e.  $(b, x) \in B^X$ , one has

$$\mathbb{E}(f | \pi^{-1} \mathcal{B}_p)(b, x) = \int_X f(b_0 \cdots b_{p-1} x') d\nu(x'). \quad (3.9)$$

In fact, the right hand side of this equation is  $\pi^{-1} \mathcal{B}_p$ -measurable, and for each  $\pi^{-1} \mathcal{B}_p$ -measurable function  $\psi$ , one has by (3.2),

$$\int_{B^X} f \psi d\beta^X = \int_B \psi(b_0, \dots, b_{p-1}) \int_X f(b_0 \cdots b_{p-1} x') d\nu(x') d\beta(b),$$

and (3.9) follows. The result is thus an immediate consequence of the Martingale convergence theorem, since, by definition, for  $\beta$ -a.e.  $b \in B$ ,  $\nu_b(f) = \mathbb{E}(f | \pi^{-1} \mathcal{B})(b)$ .

b) This equality follows from a) applied to a countable collection of functions  $f$  which generate the Borel  $\sigma$ -algebra  $\mathcal{X}$ .

c) It follows from (3.2) and (3.3) that for any bounded Borel function  $f$  on  $X$ , one has  $\nu(f) = \int_{B^X} f(x) d\beta^X(b, x) = \int_B \nu_b(f) d\beta(b)$ .

d) Let  $b \mapsto \nu'_b$  be a  $\mathcal{B}$ -measurable collection of probability measures on  $X$  satisfying the conditions. We will define the probability measure  $\lambda = \int_B \delta_b \otimes \nu'_b d\beta(b)$  on  $B^X$  and prove that  $\lambda = \beta^X$ . To this end, we

compute, for any positive  $\mathcal{B}_n \otimes \mathcal{X}$ -measurable function  $\varphi$  on  $B^X$ , using the two properties (3.7) and (3.8) for the family  $\nu'_b$  and using (3.2),

$$\begin{aligned} \lambda(\varphi) &= \int_B \int_X \varphi(b, x) d\nu'_b(x) d\beta(b) \\ &= \int_B \int_B \int_X \varphi(b_0 \cdots b_{n-1} b', b_0 \cdots b_{n-1} y) d\nu'_{b'}(y) d\beta(b') d\beta(b) \\ &= \int_B \int_X \varphi(b, b_0 \cdots b_{n-1} y) d\beta(b) d\nu(y) = \beta^X(\varphi). \end{aligned}$$

This implies  $\lambda = \beta^X$  since, by the uniqueness of disintegration, for  $\beta$ -a.e.  $b$ , one has  $\nu'_b = \nu_b$ .

e) One has

$$\mu * \nu = \int_G \int_B g_* \nu_b d\beta(b) d\mu(g) = \int_B b_{0*} \nu_{Tb} d\beta(b) = \int_B \nu_b d\beta(b) = \nu.$$

□

**Remark 3.3.** *Whenever  $X$  is a metrizable separable locally compact space and the action of  $G$  on  $X$  is continuous (this will always be the case in our applications), one then has*

$$\nu_b = \lim_{p \rightarrow \infty} b_{0*} \cdots b_{p*} \nu. \quad (3.10)$$

*This is the original introduction of the object by Furstenberg [8].*

**Remark 3.4.** *One easily shows that the probability measure  $\nu$  is  $\mu$ -ergodic if and only if the probability measure  $\beta^X$  is  $T^X$ -ergodic.*

We indicate a nice application of these constructions.

**Corollary 3.5.** *Let  $\mu$  be a probability measure on  $G$ , let  $\nu$  and  $\nu'$  be two  $\mu$ -stationary measures on two standard Borel spaces  $(X, \mathcal{X})$  and  $(X', \mathcal{X}')$ , endowed with a Borel action of  $G$ . Then, the probability measure  $\nu'' = \int_B \nu_b \otimes \nu'_b d\beta(b)$  is a  $\mu$ -stationary Borel probability measure on the product space  $X \times X'$ .*

*Proof.* In fact, the  $\mathcal{B}$ -measurable family  $b \mapsto \nu''_b = \nu_b \otimes \nu'_b$  of probability measures on  $X \times X'$  satisfies, for  $\beta$ -a.e.  $b \in B$ , the equality  $b_{0*} \nu''_{Tb} = \nu''_b$ . □

**3.3. Fibered systems over a suspension.** *The dynamical system which we will need for our problem is a fibered product over a suspension.*

Let  $M$  be a compact metrizable topological group and let  $\tau = (\tau_{\mathbb{R}}, \tau_M) : B \times \mathbb{R}_+ \times M$  a  $\mathcal{B}$ -measurable map with  $\tau_{\mathbb{R}} \neq 0$ . We denote by  $(B^\tau, \mathcal{B}^\tau, \beta^\tau, T^\tau)$  the semi-flow obtained by the suspension of



$(B, \mathcal{B}, \beta, T)$  using  $\tau$ , defined in §2.2. We will now construct a fibered semi-flow  $B^{\tau, X}$  over  $B^\tau$ .

For  $\ell \geq 0$  and for  $\beta^\tau$ -a.e.  $c = (b, k, m) \in B^\tau$ , we introduce the map  $\rho_\ell(c)$  of  $X$  given by, for any  $x \in X$ ,

$$\rho_\ell(c)x = b_{p_\ell(b,k)-1}^{-1} \cdots b_0^{-1}x,$$

and denote  $\nu_c = \nu_b$ . We then have the following equivariance property for the probability measures on  $X$ :

**Lemma 3.6.** *For  $\beta^\tau$ -a.e.  $c = (b, k, m) \in B^\tau$  and for every  $\ell \geq 0$ , one has*

$$\nu_{T_\ell^\tau c} = \rho_\ell(c)_* \nu_c.$$

*Proof.* Because of Lemma 3.2(b) and the equality  $\nu_{T_\ell^\tau c} = \nu_{T^{p_\ell(b,k)}b}$ , we have also  $\nu_c = (b_0 \cdots b_{p_\ell(b,k)-1})_* \nu_{T_\ell^\tau c}$ .  $\square$

We define the semi-flow  $(B^{\tau, X}, \mathcal{B}^{\tau, X}, \beta^{\tau, X}, T^{\tau, X})$  fibered over  $(B^\tau, \mathcal{B}^\tau, \beta^\tau, T^\tau)$  as follows. We set  $B^{\tau, X} = B^\tau \times X$  and

$$\beta^{\tau, X} = \int_{B^\tau} \delta_c \otimes \nu_c d\beta^\tau(c).$$

We denote by  $\mathcal{B}^{\tau, X}$  the completion of the product  $\sigma$ -algebra  $\mathcal{B}^\tau \otimes \mathcal{X}$  with respect to the probability measure  $\beta^{\tau, X}$  and, for  $(c, x) \in B^{\tau, X}$  and  $\ell \geq 0$ , we set

$$T_\ell^{\tau, X}(c, x) = (T_\ell^\tau c, \rho_\ell(c)x).$$

**Lemma 3.7.** *For all  $\ell \geq 0$ , the transformation  $T_\ell^{\tau, X}$  of  $B^{\tau, X}$  preserves the measure  $\beta^{\tau, X}$ .*

*Proof.* This follows from Lemmas 2.4 and 3.6.  $\square$

Denote  $\mathcal{Q}_\ell^{\tau, X} = (T_\ell^{\tau, X})^{-1}(\mathcal{B}^{\tau, X})$  and denote by  $\mathcal{Q}_\infty^{\tau, X}$  the tail  $\sigma$ -algebra of  $(B^{\tau, X}, \mathcal{B}^{\tau, X}, \beta^{\tau, X}, T^{\tau, X})$ , that is the decreasing intersection of sub- $\sigma$ -algebras  $\mathcal{Q}_\infty^{\tau, X} = \bigcap_{\ell \geq 0} \mathcal{Q}_\ell^{\tau, X}$ . Similarly, denote by  $\mathcal{Q}_\ell$  the decreasing family of  $\sigma$ -algebras  $\mathcal{Q}_\ell = (T_\ell^\tau)^{-1}(\mathcal{B}^\tau)$  and by  $c \mapsto \beta_c^\ell$  the conditional measure of  $\beta^\tau$  relative to  $\mathcal{Q}_\ell$ .

We can conclude the preceding discussion with the following corollary which is at the heart of our drift argument.

**Corollary 3.8.** *For any  $\beta^{\tau, X}$ -integrable function  $\varphi : B^{\tau, X} \rightarrow \mathbb{R}$ , for every  $\ell \geq 0$ , for  $\beta^{\tau, X}$ -a.e.  $(c, x) \in B^{\tau, X}$ , one has*

$$\mathbb{E}(\varphi | \mathcal{Q}_\ell^{\tau, X})(c, x) = \int_{B^\tau} \varphi(c', \rho_\ell(c')^{-1} \rho_\ell(c)x) d\beta_c^\ell(c'). \quad (3.11)$$

*Proof.* This follows from Lemma 2.5.  $\square$

**3.4. Measure of relative stable leaves.** *In order to be able to apply our drift argument, we will need to know that the probability measures  $\nu_b$  give no mass to the relative stable leaves of the factor map  $B^{\tau, X} \rightarrow B^\tau$ . Proposition 3.9 below will give us a useful criterion which will enable us to prove this.*

We will assume from now on that  $X$  is a locally compact metrizable topological space and that the action of  $G$  on  $X$  is continuous. We denote by  $d$  a metric on  $X$  inducing the topology. For  $(b, x)$  in  $B \times X$ , we denote by

$$W_b(x) = \{x' \in X : \lim_{p \rightarrow \infty} d(\rho_p(b)x, \rho_p(b)x') = 0\}$$

the stable leaf relative to  $(b, x)$ . This leaf does not depend on the choice of the metric  $d$  whenever  $X$  is compact, but may depend on  $d$  in general. However, in all cases, one has the following proposition. Recall that a continuous map is called *proper* if the inverse image of any compact subset is compact. Denote by  $A_\mu$  the averaging operator on  $X \times X$  given by, for any positive function  $v$  on  $X \times X$  and any  $(x, y)$  in  $X \times X$ ,

$$A_\mu(v)(x, y) = \int_G v(gx, gy) d\mu(g).$$

This operator is thus the convolution operator of the image  $\check{\mu}$  of the measure  $\mu$  under inversion  $g \mapsto g^{-1}$ . We denote by  $\Delta_X$  the diagonal in  $X \times X$ .

**Proposition 3.9.** *Suppose the following hypothesis (HC):*

*There exists a function  $v : (X \times X) \setminus \Delta \rightarrow [0, \infty)$  such that, for any compact subset  $K \subset X$ , the restriction of  $v$  to  $K \times K \setminus \Delta$  is proper and there are constants  $a \in (0, 1)$  and  $C > 0$  such that  $A_\mu(v) \leq av + C$ .*

*Let  $\nu$  be a  $\mu$ -stationary non-atomic Borel probability measure on  $X$ . Then for  $\beta^X$ -a.e.  $(b, x) \in B \times X$ , one has*

$$\nu_b(W_b(x)) = 0.$$

Hypothesis (HC) signifies that on average,  $\mu$  contracts the function  $v$  at a fixed rate.

The proof of this fact follows three steps. The first step is the most delicate, and is contained in the following lemma.

**Lemma 3.10.** *Assume hypothesis (HC), and let  $\nu$  be a  $\mu$ -stationary Borel probability measure such that, for  $\beta$ -a.e.  $b \in B$ , the probability measure  $\nu_b$  is a Dirac mass. Then  $\nu$  is a Dirac mass.*

*Proof.* Let  $\kappa : B \rightarrow X$  denote the  $\mathcal{B}$ -measurable map such that, for  $\beta$ -a.e.  $b \in B$ , one has

$$\nu_b = \delta_{\kappa(b)}. \tag{3.12}$$

The strategy will consist of studying the corresponding random walk on  $X \times X$ . Roughly speaking, the existence of  $\kappa$  and the Chacon-Ornstein ergodic theorem will ensure that this random walk approaches the diagonal  $\Delta_X$  while the existence of  $v$  pushes the random walk away from the diagonal. Here are the details.

For  $g \in G$  and  $b = (b_0, b_1, \dots) \in B$ , let  $gb = (g, b_0, b_1, \dots)$ . By Lemma 3.2(b) we have, for  $\mu$ -a.e.  $g \in G$  and  $\beta$ -a.e.  $b \in B$ ,

$$\kappa(gb) = g\kappa(b).$$

By Lemma 3.2(c), we also have the equality

$$\nu = \kappa_*\beta.$$

Endow  $B = G^{\mathbb{N}}$  with the product topology. By Lusin's theorem, for every  $\varepsilon > 0$ , there is a compact subset  $K_0 \subset B$  such that  $\beta(K_0) = 1 - \varepsilon$  and the restriction of  $\kappa$  to  $K_0$  is uniformly continuous. Denote by  $K$  the compact image  $K = \kappa(K_0)$ . Since the restriction of  $v$  to  $K \times K \setminus \Delta$  is proper, one has

$$\begin{aligned} &\forall M > 0, \exists n_M > 0, \forall n \geq n_M, \forall b, b' \in B, \\ &\forall g_1, \dots, g_n \in G \text{ such that } g_1 \cdots g_n b \in K_0 \text{ and } g_1 \cdots g_n b' \in K_0, \quad (3.13) \\ &\text{we have } v(\kappa(g_1 \cdots g_n b), \kappa(g_1 \cdots g_n b')) \geq M. \end{aligned}$$

We now introduce the transfer operator  $L_\mu$  on  $B$  given by, for each  $\varphi_0 \in L^1(B, \beta)$ , for  $\beta$ -a.e.  $b \in B$ ,

$$(L_\mu \varphi_0)(b) = \int_G \varphi_0(gb) d\mu(g).$$

Since it is the adjoint of the shift  $T$ ,  $L_\mu$  is an ergodic operator. The theorem of Chacon-Ornstein [4], applied to the function  $\varphi_0 = 1_{K_0}$ , ensures that for  $b$  outside a subset  $N \subset B$  of zero measure, we have the equality

$$\lim_{p \rightarrow \infty} \frac{1}{p} \sum_{0 \leq n \leq p} (L_\mu^n 1_{K_0})(b) = \beta(K_0) = 1 - \varepsilon. \quad (3.14)$$

By possibly increasing the set  $N$ , we may also assume that for any  $b \in B \setminus N$ , for any integer  $n \geq 0$ , and for  $\mu^{\otimes n}$ -a.e.  $(g_1, \dots, g_n) \in G^n$ , one has  $\kappa(g_1 \cdots g_n b) = g_1 \cdots g_n \kappa(b)$ .

Suppose by contradiction that  $\nu$  is not a Dirac mass. Then the set

$$E = \{(b, b') \in B \times B : \kappa(b) \neq \kappa(b')\}$$

is of positive measure with respect to  $\beta \otimes \beta$ . Therefore we can find points  $b_0$  and  $b'_0$  outside of  $N$  such that

$$\kappa(b_0) \neq \kappa(b'_0). \quad (3.15)$$

We now use condition (HC). It implies that for all  $n \geq 0$ , one has

$$A_\mu^n v \leq a^n v + (1 + \dots + a^{n-1})C.$$

For every  $x \neq x' \in X$ , we deduce the upper bound

$$\frac{1}{p} \sum_{0 \leq n \leq p} (A_\mu^n v)(x, x') \leq \frac{1}{p(1-a)} v(x, x') + \frac{1}{1-a} C. \quad (3.16)$$

We will now apply this upper bound to the points  $x = \kappa(b_0)$  and  $x' = \kappa(b'_0)$ . Fix  $M > 0$ . Note that, thanks to (3.14), there exists an integer  $p_0 \geq n_M$  such that for all  $p \geq p_0$ ,

$$\frac{1}{p} \sum_{0 \leq n \leq p} (L_\mu^n 1_{K_0})(b_0) \geq 1 - 2\varepsilon \text{ and } \frac{1}{p} \sum_{0 \leq n \leq p} (L_\mu^n 1_{K_0})(b'_0) \geq 1 - 2\varepsilon.$$

As a consequence,

$$\frac{1}{p} \sum_{0 \leq n \leq p} (A_\mu^n v)(\kappa(b_0), \kappa(b'_0)) \geq \left(1 - 4\varepsilon - \frac{p_0}{p}\right) M.$$

Taking a limit as  $p \rightarrow \infty$  and using (3.16) we obtain

$$(1 - 4\varepsilon)M \leq C/(1 - a).$$

Since  $M$  was arbitrary, we get a contradiction as soon as  $\varepsilon < 1/4$ . Therefore  $\nu$  is a Dirac mass.  $\square$

The second step is the following lemma:

**Lemma 3.11.** *Under assumption (HC), let  $\nu$  be a non-atomic  $\mu$ -stationary probability measure on  $X$ . Then for  $\beta$ -a.e.  $b \in B$ , the probability measure  $\nu_b$  is non-atomic.*

*Proof.* The strategy consists, after several reductions involving the ergodicity of  $\beta$ , in constructing a stationary probability measure on a space  $Y$  on which one can apply Lemma 3.10.

Suppose by contradiction that the set  $D = \{b \in B : \nu_b \text{ has atoms}\}$  is of positive measure. Since  $\nu_b = b_{0*} \nu_{Tb}$ , the set  $D$  is  $T$ -invariant. Since  $\beta$  is  $T$ -ergodic, this means that  $\beta(D) = 1$ . The same argument also shows that the maximal mass  $M_b$  of an atom of  $\nu_b$  is a  $\beta$ -almost surely constant function and that the number  $N_b$  of atoms whose  $\nu_b$  measure is  $m_b$  is also a.e. constant. We denote this mass by  $m_0$  and this number of atoms by  $N_0$ . Denote by  $\nu'_b$  the probability measure with  $N_0$  atoms of  $\nu_b$  each of mass  $m_0$ . We also have the equality  $\nu'_b = b_{0*} \nu'_{Tb}$ . By Lemma 3.2(e), the probability measure  $\nu' = \int_B \nu'_b d\beta(b)$  on  $X$  is also  $\mu$ -stationary and one can write  $\nu$  as the sum of  $m_0 \nu'$  and a stationary measure of mass  $(1 - m_0)$ . By assumption,  $\nu'$  is also non-atomic, and

by Lemma 3.2(d), the measures  $\nu'_b$  are the limit measures of  $\nu'$ , and thus we can henceforth assume that  $\nu = \nu'$ .

Let  $S_{N_0}$  denote the group of permutations of  $\{1, \dots, N_0\}$  and let  $Y$  denote the quotient  $X^{N_0}/S_{N_0}$  and  $p : X^{N_0} \rightarrow Y$  the projection. The group  $G$  acts naturally on  $Y$ . We check that  $Y$  satisfies hypothesis (HC). Let  $v$  denote the function and let  $a, C$  denote the constants which appear in hypothesis (HC) for  $X$  and introduce the map  $w : Y \times Y \setminus \Delta \rightarrow [0, \infty)$  given, for  $y = p(x_1, \dots, x_{N_0})$  and  $y' = p(x'_1, \dots, x'_{N_0})$  with  $x_i, x'_i \in X$ , by

$$w(y, y') = \sum_{\sigma \in S_{N_0}} \min_{1 \leq i \leq N_0} v(x_i, x'_{\sigma(i)}).$$

This map  $w$  is certainly continuous and proper on  $K \times K \setminus \Delta$  for any compact subset  $K \subset Y$ . It also satisfies an upper bound

$$A_\mu(w) \leq aw + CN_0!.$$

Introduce the family  $b \mapsto \nu''_b = p_*(\nu_b^{\otimes N_0})$  of probability measures on  $Y$ . We also have the equality  $\nu''_b = b_{0*}\nu''_{Tb}$ . By Lemma 3.2(e), the probability measure  $\nu'' = \int_B p_*(\nu_b^{\otimes N_0}) d\beta(b)$  is  $\mu$ -stationary. By construction, for  $\beta$ -a.e.  $b \in B$  the measure  $\nu''_b$  is a Dirac mass. Lemma 3.10 then shows that  $\nu''$  is also a Dirac mass  $\delta_{y_0}$ . Therefore, for  $\beta$ -a.e.  $b \in B$ ,  $\nu''_b = \delta_{y_0}$  and hence  $\nu$  is of finite support, a contradiction.  $\square$

The last step does not use assumption (HC).

**Lemma 3.12.** *Let  $\nu$  be a  $\mu$ -stationary probability measure on  $X$  such that, for  $\beta$ -a.e.  $b \in B$ , the measure  $\nu_b$  is non-atomic. Then for  $\beta^X$ -a.e.  $(b, x) \in B \times X$ ,  $\nu_b(W_b(x)) = 0$ .*

*Proof.* Consider the transformation on  $B \times X \times X$  given by, for  $(b, x, x') \in B \times X \times X$ ,

$$R(b, x, x') = (Tb, b_0^{-1}x, b_0^{-1}x').$$

Lemma 3.1 and Corollary 3.5 show that  $R$  preserves the probability measure

$$\Lambda = \int_B \delta_b \otimes \nu_b \otimes \nu_b d\beta(b).$$

Denote

$$Z = \{(b, x, x') \in B \times X \times X : \lim_{p \rightarrow \infty} d(\rho_p(b)x, \rho_p(b)x') = 0\}$$

and, for  $(b, x, x') \in B \times X \times X$ , write  $\varphi(b, x, x') = d(x, x')$ . By assumption, for  $\beta$ -a.e.  $b$ , the measure  $\nu_b$  is non-atomic, and hence  $\nu_b \otimes \nu_b$  gives no mass to the diagonal  $X \times X$ . Therefore the function  $\varphi$  is  $\Lambda$ -a.e. nonzero. By construction, for  $\Lambda$ -a.e.  $z \in Z$ , one has

$\lim_{p \rightarrow \infty} \varphi(R^p(z)) = 0$  and thus, by the Poincaré recurrence theorem,  $\Lambda(Z) = 0$ , as required.  $\square$

*Proof of Proposition 3.9.* Follows from Lemma 3.11 and 3.12.  $\square$

#### 4. CONDITIONAL MEASURES

*In this chapter we collect certain properties of conditional measures of a probability measure for a Borel action of a locally compact group.*

**4.1. Conditional measures.** *We recall the construction of conditional measures.*

Let  $R$  be a locally compact separable metrizable group and  $(Z, \mathcal{Z})$  a standard Borel space with a Borel action of  $R$ . Let  $\lambda$  be a Borel probability measure on  $Z$ . Suppose that the stabilizer subgroups for the action of  $R$  on  $Z$  are discrete. We will now explain how the action of  $R$  on  $Z$  makes it possible to ‘disintegrate the measure  $\lambda$  along  $R$ -orbits’, to obtain measures on  $R$  which are unique up to normalization. More precisely:

Let  $\mathcal{M}(R)$  denote the space of positive nonzero Radon measures on  $R$  and let  $\mathcal{M}_1(R) = \mathcal{M}(R)/\simeq$  be the space of such measures up to scaling: two Radon measures  $\sigma_1, \sigma_2$  are called equal up to scaling, and we write  $\sigma_1 \simeq \sigma_2$ , if there is  $c > 0$  such that  $\sigma_2 = c\sigma_1$ . We can choose a representative of each equivalence class: we fix an increasing sequence of compact subsets  $(K_n)$  of  $R$  which cover  $R$  and choose  $\sigma$  so that  $\sigma(K_n) = 1$ , where  $n$  is the smallest  $m$  for which  $\sigma(K_m) > 0$ .

We say that a Borel subset  $\Sigma \subset Z$  is a *discrete section* of the action of  $R$  if, for any  $z \in Z$ , the set of visit times  $\{r \in R : rz \in \Sigma\}$  is discrete and closed in  $R$ . The main theorem of [12] shows that there is a discrete section  $\Sigma$  for the action of  $R$  such that  $R\Sigma = Z$ .

We choose a discrete section  $\Sigma$  for the action of  $Z$  on  $R$  and denote  $a : R \times \Sigma \rightarrow Z, (r, z) \mapsto rz$ . The measure  $a^*\lambda$  on  $R \times \Sigma$  defined, for any positive Borel function  $f$  on  $R \times \Sigma$ , by

$$a^*\lambda(f) = \int_Z \left( \sum_{(r, z') \in a^{-1}(z)} f(r, z') \right) d\lambda(z), \quad (4.1)$$

is a  $\sigma$ -finite Borel measure on  $R \times \Sigma$ . This follows from the fact that for any compact subset  $C \subset R$ , and any  $z \in Z$ , the set  $(C \times \Sigma) \cap a^{-1}(z)$  is finite.

We denote  $\pi_\Sigma : R \times \Sigma \rightarrow \Sigma$  the projection on the second factor, and by  $\lambda_\Sigma$  the image under  $\pi_\Sigma$  of a finite measure on  $R \times \Sigma$  equivalent to

$a^*\lambda$ . We therefore have, for any positive Borel function on  $R \times Z$ ,

$$a^*\lambda(f) = \int_{\Sigma} \int_R f(r, z) d\sigma_{\Sigma}(z)(r) d\lambda_{\Sigma}(z). \quad (4.2)$$

Note that the conditional measures  $\sigma_{\Sigma}(z)$  are also Radon measures on  $R$ . This results once more from the finiteness of the sets  $(C \times \Sigma) \cap a^{-1}(z)$ .

We denote by  $t_r$  the right-translation by an element  $r \in R$ .

**Lemma 4.1.** *Let  $\Sigma$  be a discrete section for the action of  $R$  on  $Z$ . For  $\lambda_{\Sigma}$ -a.e.  $z \in \Sigma$ , for all  $r \in R$  such that  $rz \in \Sigma$ , we have*

$$\sigma_{\Sigma}(z) \simeq t_{r*}\sigma_{\Sigma}(rz).$$

*Proof.* The difficulty comes from the fact that one wants this condition to be satisfied for an uncountable family of elements  $r \in R$ . To deal with this difficulty, it suffices to remark that there is a countable family, indexed by  $i \in \mathbb{N}$ , of Borel sets  $\Sigma_i \subset \Sigma$ , and Borel maps  $r_i : \Sigma_i \rightarrow R$ , such that

$$\{(z, r) \in \Sigma \times R : rz \in \Sigma\} = \bigcup_{i \in \mathbb{N}} \{(z, r_i(z)) : z \in \Sigma_i\},$$

and such that, for  $\lambda_{\Sigma}$ -a.e.  $z \in \Sigma_i$ ,  $\sigma_{\Sigma}(z) \simeq t_{r_i(z)*}\sigma_{\Sigma}(r_i(z)z)$ .  $\square$

**Proposition 4.2.** *Consider a Borel action with discrete stabilizers of a locally compact separable metrizable group  $R$  on a standard Borel space  $(Z, \mathcal{Z})$ .*

*Then there is a Borel map  $\sigma : Z \rightarrow \mathcal{M}_1(R)$  and a Borel subset  $E \subset Z$  such that  $\lambda(Z \setminus E) = 0$  and such that, for any discrete section  $\Sigma \subset Z$  for the action of  $R$ , for  $\lambda_{\Sigma}$ -a.e.  $z_0 \in \Sigma$ , for every  $r \in R$  such that  $rz_0 \in E$ ,*

$$\sigma(z_0) \simeq t_{r*}\sigma_{\Sigma}(rz_0).$$

*This map  $\sigma$  is unique up to a set of  $\lambda$ -measure zero.*

*For every  $r \in R$  and every  $z \in E$  such that  $rz \in E$ , we have*

$$\sigma(z) \simeq t_{r*}(\sigma(rz)). \quad (4.3)$$

The measure  $\sigma(z)$  is called the conditional measure of  $z$  along the action of  $R$ .

*Proof.* We choose a discrete section  $\Sigma_0$  such that  $R\Sigma_0 = Z$ . By Lemma 4.1, for  $\lambda$ -a.e.  $z \in Z$ , if one writes  $z = rz_0$  with  $r \in R$  and  $z_0 \in \Sigma_0$ , the measure  $\sigma(z) = t_{r*}^{-1}\sigma_{\Sigma_0}(z_0) \in \mathcal{M}_1(R)$  does not depend on choices, i.e. different choices of  $z_0$  only affect it by rescaling.

This defines the map  $\sigma$ . The asserted property of  $\sigma$  follows from the Lemma applied to  $\Sigma \cup \Sigma_0$  which is also a discrete section. Assertion (4.3) follows. Uniqueness of  $\sigma$  is clear.  $\square$

The use of conditional measures in geometric ergodic theory is based, among others, on the work of Ledrappier-Young. Its use in problems of measure classification on homogeneous spaces has appeared in [3] and earlier in work of Katok and Spatzier.

**4.2. Disintegration along stabilizers.** *In this section we explain how to exploit the invariance properties under translation, of conditional measures along an action.*

Denote by  $\text{Gr}(\mathbb{R}^d)$  the Grassmannian variety of  $\mathbb{R}^d$ . The following proposition asserts that the disintegration of  $\lambda$  to conditional measures along the stabilizer gives probability measures invariant under the stabilizer. In a topological group  $S$ , we denote by  $S_0$  the connected component of the identity.

**Proposition 4.3.** *Let  $(Z, \mathcal{Z})$  be a standard Borel space endowed with a Borel action of  $\mathbb{R}^d$  with discrete stabilizers, and let  $\lambda$  be a Borel probability measure on  $Z$ . For  $\lambda$ -a.e.  $z \in Z$ , we denote by  $\sigma(z)$  the conditional measure of  $z$  for the action of  $\mathbb{R}^d$ , and*

$$V_z = \{r \in \mathbb{R}^d : t_{r*}\sigma(z) = \sigma(z)\}_0,$$

and by

$$\lambda = \int_Z \lambda_z d\lambda(z)$$

the disintegration of  $\lambda$  along the map  $Z \rightarrow \text{Gr}(\mathbb{R}^d)$ ,  $z \mapsto V_z$ . Then for  $\lambda$ -a.e.  $z \in Z$ , the probability measure  $\lambda_z$  is  $V_z$ -invariant.

This proposition is a consequence of the following three lemmas. The first one uses notation which are different from those used in Proposition 4.3.

**Lemma 4.4.** *Let  $(Z, \mathcal{Z}, \lambda)$  be a Lebesgue space,  $(Y, \mathcal{Y})$  a standard Borel space equipped with a Borel action of  $\mathbb{R}^d$ ,  $f : Z \rightarrow Y$  a measurable map and  $I : Z \rightarrow \text{Gr}(\mathbb{R}^d)$  a measurable map such that for  $\lambda$ -a.e.  $z \in Z$ ,  $I(z)$  stabilizes  $f(z)$ .*

*Denote by  $\lambda = \int_Z \lambda_z d\lambda(z)$  the disintegration of  $\lambda$  along  $I$ .*

*Then for  $\lambda$ -a.e.  $z \in Z$ , for  $\lambda_z$ -a.e.  $z' \in Z$ , the element  $f(z')$  is  $I(z)$ -invariant.*

*Proof.* In fact, for  $\lambda$ -a.e.  $z \in Z$ , for  $\lambda_z$ -a.e.  $z' \in Z$ , we have from the definition of conditional measures, that  $I(z) = I(z')$  and hence, by assumption,  $f(z')$  is  $I(z')$ -invariant.  $\square$

The second lemma uses once more the notation of Proposition 4.3.



**Lemma 4.5.** *Let  $(Z, \mathcal{Z})$  be a standard Borel space equipped with a Borel action of  $\mathbb{R}^d$  with discrete stabilizers, and let  $\lambda$  be a Borel probability measure on  $Z$ . Let  $(Y_0, \mathcal{Y}_0)$  be a standard Borel space and let  $\varphi : Z \rightarrow Y_0$  be a measurable map for which there exists a subset  $E \subset Z$  such that  $\lambda(Z \setminus E) = 0$  and for every  $z \in E$  and  $r \in \mathbb{R}^d$  with  $rz \in E$ , we have  $\varphi(z) = \varphi(rz)$ . Denote  $z \mapsto \sigma(z) \in \mathcal{M}(\mathbb{R}^d)$  the conditional measure at  $z$  of  $\lambda$  along the  $\mathbb{R}^d$ -orbits, and denote  $\lambda = \int_Z \lambda_z d\lambda(z)$  the disintegration of  $\lambda$  along  $\varphi$ . Then, for  $\lambda$ -a.e.  $z \in Z$ , for  $\lambda_z$ -a.e.  $z' \in Z$ ,  $\sigma(z')$  is also the conditional measure of  $z'$  of  $\lambda_z$  along the action of  $\mathbb{R}^d$ .*

*Proof.* We adapt the argument of transitivity of the disintegration of measures in this context.

Recall the gist of the argument in the classical context: we are given a Lebesgue space  $(A, \mathcal{A}, \alpha)$ , and two standard Borel spaces  $(B, \mathcal{B}), (C, \mathcal{C})$  along with measurable maps  $f : A \rightarrow B$  and  $g : B \rightarrow C$ . Then almost surely, the conditional measures of  $\alpha$  along  $f$  coincide with the conditionals along  $f$  of the conditionals of  $\alpha$  along  $g \circ f$ . More precisely, denote  $\alpha = \int_A \alpha_a d\alpha(a)$  and  $\alpha = \int_A \alpha_{a'} d\beta_a(a')$ , the disintegrations of  $\alpha$  respectively along  $f$  and along  $g \circ f$ . We then have, for  $\alpha$ -a.e.  $a$ , the equality  $\beta_a = \int_A \alpha_{a'} d\beta_a(a')$  which gives the disintegration of  $\beta_a$  along  $f$ .  $\square$

**Lemma 4.6.** *Let  $(Z, \mathcal{Z})$  be a standard Borel space, equipped with a Borel action of  $\mathbb{R}^d$  with discrete stabilizers,  $W$  a linear subspace of  $\mathbb{R}^d$ ,  $\lambda$  a probability measure on  $(Z, \mathcal{Z})$ , and  $z \mapsto \sigma(z) \in \mathcal{M}(\mathbb{R}^d)$  the conditional measures at  $z$  of  $\lambda$  along the action of  $\mathbb{R}^d$ . Suppose that for  $\lambda$ -a.e.  $z \in Z$ ,  $\sigma(z)$  is invariant under translations by  $W$ . Then  $\lambda$  is also invariant under the action of  $W$ .*

*Proof.* As in §4.1, denote by  $\Sigma$  a discrete section for the action of  $\mathbb{R}^d$  such that  $\mathbb{R}^d \Sigma = Z$  and let  $a$  be the map  $a : \mathbb{R}^d \times \Sigma \rightarrow Z$ ,  $(r, z) \mapsto rz$ . By assumption the measure  $a^* \lambda$  is  $W$ -invariant, and hence so is the measure  $\lambda$ .  $\square$

*Proof of Proposition 4.3.* Applying Lemma 4.4 with  $Y = \mathcal{M}(\mathbb{R}^d)$ ,  $f = \sigma$  and  $I(z) = V_z$ , and then Lemma 4.5 with  $Y_0 = \text{Gr}(\mathbb{R}^d)$  and  $\varphi(z) = V_z$ . We find that, for  $\lambda$ -a.e.  $z \in Z$ , for  $\lambda_z$  a.e.  $z' \in Z$ , the conditional measure  $\sigma_z(z')$  of  $\lambda_z$  for the action of  $\mathbb{R}^d$  on  $Z$  is  $V_z$ -invariant and hence, by Lemma 4.6, that the measure  $\lambda_z$  is  $V_z$ -invariant.  $\square$

## 5. RANDOM WALKS ON LIE GROUPS

*In this chapter we introduce, for a strongly irreducible random walk, a dynamical system  $(B^\tau, \mathcal{B}^\tau, \beta^\tau, T^\tau)$  which is a suspension of the Bernoulli*

system  $(B, \mathcal{B}, \beta, T)$ . We then study the asymptotic behavior of the random walk in order to be able to control the drift in §7.1.

**5.1. Stationary measures on the flag variety.** Let  $G$  be a real semisimple virtually connected Lie group, that is it has a finite number of connected components.

**Definition 5.1.** We say that a Borel probability measure on  $G$  is Zariski dense if the semigroup  $\Gamma_\mu$  generated by the support of  $\mu$  has a Zariski dense image in the adjoint group  $\text{Ad}(G) \subset \text{GL}(\mathfrak{g})$ .

Let  $\mu$  be a Zariski dense probability measure on  $G$  with compact support. We also denote by  $(B, \mathcal{B}, \beta, T)$  the two-sided Bernoulli shift on the alphabet  $(G, \mathcal{G}, \mu)$ , where  $\mathcal{G}$  denotes the Borel  $\sigma$ -algebra of  $G$ .

Let  $P \subset G$  be a minimal parabolic subgroup. Write  $P = ZU$ , where  $U$  is the unipotent radical of  $P$  and  $Z$  is a maximal reductive subgroup of  $P$ . Denote by  $A$  the Cartan subgroup of  $Z$  and by  $A^+$  the Weyl chamber of  $A$  associated with an order corresponding to the choice of  $P$ . Choose a Cartan involution of  $G$  which leaves  $Z$  invariant and denote by  $K$  the maximal compact subgroup of  $G$  consisting of points fixed by this Cartan involution.

Let  $V$  be a real representation of  $G$  of dimension  $d$  which is strongly irreducible, that is, its restriction to the connected component of the identity in  $G$  is also irreducible. Fix once and for all a  $K$ -invariant Euclidean norm  $\|\cdot\|$  on  $V$  such that the elements of  $A$  act on  $V$  in a symmetric fashion.

Denote by  $\chi$  the largest weight for  $A$  in  $V$ , let  $V_0 = V_\chi$  be the corresponding weight space in  $V$ , so that  $PV_0 \subset V_0$ , and let  $d_0 = \dim V_0$ . Denote by  $V'_0$  the subspace of  $V$  which is the sum of the other weight-subspaces, so that  $V = V_0 \oplus V'_0$ .

The following proposition is essentially due to Furstenberg and Kesten [10]. Denote by  $\text{Gr}_{d_0}(V)$  the Grassmannian variety of  $d_0$ -planes in  $V$ .

**Proposition 5.2.** There are  $\mathcal{B}$ -measurable maps  $B \rightarrow \text{Gr}_{d_0}(V)$ ,  $b \mapsto V_b$  and  $B \rightarrow \text{Gr}_{d-d_0}(V)$ ,  $b \mapsto V'_b$ , such that:

- a) For  $\beta$ -a.e.  $b \in B$ , any accumulation point  $m$  of the sequence  $\left( \frac{b_0 \cdots b_n}{\|b_0 \cdots b_n\|} \right)_n$ , has as its image  $\text{Im}(m) = V_b$  and is an isometry on  $\ker(m)^\perp$ .
- b) For  $\beta$ -a.e.  $b \in B$ , any accumulation point  $m'$  of the sequence  $\left( \frac{b_n \cdots b_0}{\|b_n \cdots b_0\|} \right)_n$ , has  $\ker(m') = V'_b$  and is an isometry on  $\ker(m')^\perp$ .
- c) For any hyperplane  $W \subset V$ , we have  $\beta(\{b \in B : V_b \subset W\}) = 0$ .
- d) For any nonzero  $v \in V$ , we have  $\beta(\{b \in B : v \in V'_b\}) = 0$ .

- e) For any  $W \in \text{Gr}_{d_0}(V)$ , we have  $\beta(\{b \in B : W \cap V'_b \neq 0\}) = 0$ .
- f) For  $\beta$ -a.e.  $b \in B$ , the limit  $\lambda_1 = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|b_0 \cdots b_n\|$  exists and is positive.

*Proof.* For a), c), and f) see [8] and [1]. The fact that the accumulation points  $m$  are of rank  $d_0 = \dim V_0$  is due to Goldsheid and Margulis [11]. It can also be deduced from the existence of loxodromic elements in  $\Gamma_\mu$ . The fact that the restriction of  $m$  to the orthocomplement of its kernel is a similarity is valid for any matrix  $\pi$  of rank  $d_0$  in the closure  $\overline{\mathbb{R}_* G} \subset \text{End}(V)$ . One easily verifies this assertion thanks to the Cartan decomposition  $G = KA^+K$ .

Assertions b) and d) are deduced from assertions a) and c) by passing to the dual representation.

Assertion e) is deduced from d) by passing to an irreducible subrepresentation of the representation of  $G$  on  $\bigwedge^{d_0} V$  generated by the line of highest weight  $\bigwedge^{d_0} V_0$ .  $\square$

When applying Proposition 5.2(a) to a suitable representation, one shows that there is a unique  $\mathcal{B}$ -measurable map  $\xi : B \rightarrow G/P$  such that, for  $\beta$ -a.e.  $b \in B$ ,

$$\xi(b) = b_0 \xi(Tb).$$

The image measure  $\xi_* \beta$  is therefore the unique  $\mu$ -stationary measure on  $G/P$ .

**Remark 5.3.** Certainly the spaces  $V_b$  and  $V'_b$  of Proposition 5.2 depend on the boundary map  $\xi$ . For  $b \in B$ , we denote by  $\check{b}$  the element  $\check{b} = (b_0^{-1}, b_1^{-1}, \dots)$  of  $B$ . For  $\beta$ -a.e.  $b \in B$ , we have  $V_b = \xi(b)V_0$  and  $V'_b = \xi(\check{b})V'_0$ . We also have  $V_b = b_0 V_{Tb}$  and  $V'_b = b_0 V'_{Tb}$ .

**5.2. The dynamical system  $B^\tau$ .** We want to construct an  $\mathbb{R} \times M$ -suspension  $(B^\tau, T^\tau)$  of the Bernoulli shift associated to  $\mu$  which enables us to estimate the asymptotic behavior of the induced random walk in an irreducible representation of  $G$ . We initially construct a function  $\theta : B \rightarrow Z$ .

Let  $s : G/P \rightarrow G/U$  be a Borel section of the projection  $G/U \rightarrow G/P$ . In practice, for constructing such a section, one can utilize Iwasawa decomposition or Bruhat decomposition. An explicit formula for  $s$  is not very important for us, because our constructions will not depend on the choice of the section  $s$ . However, for simplicity, suppose that the section is constructed with the aid of Iwasawa decomposition. More precisely, write  $M = Z \cap K$ . The Iwasawa decomposition  $G = KP$  makes it possible to choose a section  $s$  such that, for every  $k \in K$ ,

$$s(kP) = km(k)U \text{ with } m(k) \in M. \quad (5.1)$$

We will say from now on that the function  $s$  has *values in  $K \bmod M$* . The group  $Z$  acts by right multiplication on  $G/U$ .

We denote by  $\sigma : G \times G/P \rightarrow Z$  the Borel cocycle given by, for every  $g \in G$  and  $x \in G/P$ ,

$$gs(x) = s(gx)\sigma(g, x).$$

We denote by  $\theta : B \rightarrow Z$  the  $\mathcal{B}$ -measurable map given by, for  $\beta$ -a.e.  $b \in B$ ,

$$\theta(b) = \sigma(b_0, \xi(Tb)).$$

We introduce the bounded function  $\theta_{\mathbb{R}} : B \rightarrow \mathbb{R}$  given, for  $\beta$ -a.e.  $b \in B$ , by

$$\theta_{\mathbb{R}}(b) = \log |\chi(\theta(b))|. \quad (5.2)$$

We will use the Furstenberg formula for the first Lyapunov exponent

$$\lambda_1 = \int_B \theta_{\mathbb{R}}(b) d\beta(b) \quad (5.3)$$

(see [8], see also [7, Thm. 1.8]), and the positivity of the first Lyapunov exponent (Proposition 5.2(f)). We then have, by Lemma 2.1, two bounded  $\mathcal{B}$ -measurable functions  $\tau_{\mathbb{R}} : B \rightarrow \mathbb{R}_+^*$  and  $\varphi : B \rightarrow \mathbb{R}$  such that

$$\theta_{\mathbb{R}} = \tau_{\mathbb{R}} + \varphi \circ T - \varphi. \quad (5.4)$$

Denote by  $\theta_M(b)$  the  $M$ -component of  $\theta(b)$ , and  $\tau_M(b) = \theta_M(b)^{-1}$  and

$$\tau = (\tau_{\mathbb{R}}, \tau_M) : B \rightarrow \mathbb{R} \times M. \quad (5.5)$$

It is the suspension  $B^{\tau}$  associated with this function  $\tau$  which we will use below.

This suspension allows us to control the norm of the words which appear in the formulas for the conditional measures, thanks to the following lemma.

**Lemma 5.4.** *For  $\beta$ -a.e.  $b \in B$ , for every  $w \in V_b$ , we have*

$$\|b_0^{-1}w\| = e^{-\theta_{\mathbb{R}}(b)}\|w\|. \quad (5.6)$$

*Proof.* By the definition of  $\theta$ , for  $\beta$ -a.e.  $b \in B$ , we have

$$b_0s(\xi(Tb)) = s(\xi(b))\theta(b).$$

Since  $w$  is in  $V_b$ , we can write  $w = s(\xi(b))v$  with  $v \in V_0$ . We note that this expression makes sense because  $U$  acts trivially on  $V_0$ . Since the norm is  $K$ -invariant, we have

$$\|b_0^{-1}w\| = \|b_0^{-1}s(\xi(b))v\| = \|\theta(b)^{-1}v\| = e^{-\theta_{\mathbb{R}}(b)}\|v\| = e^{-\theta_{\mathbb{R}}(b)}\|w\|.$$

□

**5.3. Behavior of random walks.** *We continue our study of the asymptotic behavior of the random walk on  $G$ .*

We will use Proposition 5.2 to control the drift in Lemma 7.3, in the form of the following Corollary.

**Corollary 5.5.** a) *For any  $\alpha > 0$ , there are  $r_0 \geq 1$ ,  $q_0 \geq 1$ , such that for any  $v \in V \setminus \{0\}$ , we have*

$$\beta \left\{ a \in B : \forall q \geq q_0, \|a_q \cdots a_0 v\| \geq \frac{1}{r_0} \|a_q \cdots a_0\| \|v\| \right\} \geq 1 - \alpha.$$

b) *For every  $\alpha > 0$  and  $\eta > 0$ , there exists  $q_0 \geq 1$ , such that, for every  $v \in V \setminus \{0\}$ , and every  $W \in \text{Gr}_{d_0}(V)$ , we have*

$$\beta \{ a \in B : \forall q \geq q_0, d(\mathbb{R}a_q \cdots a_0 v, a_q \cdots a_0 W) \leq \eta \} \geq 1 - \alpha.$$

In order to prove the corollary, we will need the following lemma in linear algebra. Denote

$$O_{d_0}(V) = \{ \pi \in \text{End}(V) : \text{rank}(\pi) = d_0 \text{ and } \pi|_{(\ker \pi)^\perp} \text{ is an isometry} \}.$$

This is a compact subset of  $\text{End}(V)$ .

**Lemma 5.6.** a) *For any  $\varepsilon > 0$ , there are  $r_0 \geq 1$ ,  $\varepsilon' > 0$  such that, for any  $g \in \text{GL}(V)$  and  $\pi \in O_{d_0}(V)$  with  $\|g - \pi\| < \varepsilon'$ , for any  $v \in V \setminus \{0\}$  with  $d(\mathbb{R}v, \ker \pi) \geq \varepsilon$  we have  $\|gv\| \geq \frac{1}{r_0} \|v\|$ .*

b) *For any  $\varepsilon > 0$  and  $\eta > 0$ , there is  $\varepsilon' > 0$  such that, for every  $g \in \text{GL}(V)$  and  $\pi \in O_{d_0}(V)$  with  $\|g - \pi\| \leq \varepsilon'$  we have, for all  $v \in V \setminus \{0\}$  and  $W \in \text{Gr}_{d_0}(V)$ , if  $d(\mathbb{R}v, \ker \pi) \geq \varepsilon$  and  $\inf_{w \in W \setminus \{0\}} d(\mathbb{R}w, \ker \pi) \geq \varepsilon$ , then  $d(\mathbb{R}gv, gW) \leq \eta$ .*

*Proof.* a). Otherwise, we can find sequences  $\pi_n$  in  $O_{d_0}(V)$ ,  $g_n \in \text{GL}(V)$  and  $v_n \in V$  with  $\|v_n\| = 1$ , such that  $\|g_n - \pi_n\| \rightarrow 0$ ,  $d(\mathbb{R}v_n, \ker \pi_n) \geq \varepsilon$  and  $\|g_n v_n\| \rightarrow 0$ . By compactness, we can assume by passing to subsequences that the  $\pi_n$  converge to  $\pi \in O_{d_0}(V)$  and  $v_n$  converge to  $v \in V$ ,  $\|v\| = 1$ . Our assertions imply that  $v$  is simultaneously in  $\ker \pi$  and is of distance at least  $\varepsilon$  from  $\ker \pi$ , a contradiction.

b). The argument is similar to the one used for proving a).  $\square$

*Proof of Corollary 5.5.* a) By Proposition 5.2(d), for any  $\alpha > 0$ , there is  $\varepsilon > 0$  such that for any  $v \in V \setminus \{0\}$ ,

$$\beta \{ a \in B : d(\mathbb{R}v, V'_a) \geq \varepsilon \} \geq 1 - \alpha/2.$$

On the other hand, by Proposition 5.2(b), for any  $\varepsilon' > 0$ , there is  $q_0 \geq 1$  such that

$$\beta \left\{ a \in B : \forall q \geq q_0, d \left( \frac{a_q \cdots a_0}{\|a_q \cdots a_0\|}, O_{d_0}(V) \right) < \varepsilon' \right\} \geq 1 - \alpha/2.$$

It now suffices to apply Lemma 5.6(a).

b) By Proposition 5.2(e), for any  $\alpha > 0$ , there is  $\varepsilon > 0$  such that for  $W \in \text{Gr}_{d_0}(V)$ ,

$$\beta \left\{ a \in B : \inf_{w \in W \setminus \{0\}} d(\mathbb{R}w, V'_b) \geq \varepsilon \right\} \geq 1 - \alpha/2.$$

It suffices to apply, as above, Proposition 5.2(b) and Lemma 5.6(b).  $\square$

## 6. HOMOGENEOUS SPACES OF SEMI-SIMPLE GROUPS

*This chapter collects diverse ergodic properties of the random walk on homogeneous spaces. These properties will enable us in §7 to develop the exponential drift argument.*

**6.1. Notations.** For the proofs of Theorem 1.1 and 1.3 we will use the same method, and common notation.

WE KEEP THE FOLLOWING NOTATION FOR THE REST OF THE PAPER.

**In the first case**, i.e. the case of Theorem 1.1,  $G$  is a connected almost-simple Lie group and  $\Lambda$  is a lattice in  $G$ . We denote by  $X$  the quotient  $G/\Lambda$  and by  $R$  the Adjoint representation of  $G$  on  $V = \mathfrak{g}$ , the Lie algebra of  $G$ .

**In the second case**, i.e. the case of Theorem 1.1,  $G$  is the Zariski closure of  $\Gamma_\mu$  in  $\text{SL}_d(\mathbb{R})$ . We denote by  $X$  the torus  $\mathbb{T}^d$  and by  $R$  the representation of  $G$  on  $V = \mathbb{R}^d$ , that is, the natural action by matrix multiplication, which we can think of as the Lie algebra of  $\mathbb{T}^d$ .

**In both cases**,  $G$  is a semisimple Lie group (we will give more details about this in Lemma 8.5), the representation  $R$  of  $G$  on  $V$  is strongly irreducible,  $\mu$  is a compactly supported probability measure such that the subsemigroup  $\Gamma = \Gamma_\mu$  generated by  $\text{supp } \mu$  is Zariski dense in  $G$ ,  $\nu$  is a non-atomic  $\mu$ -stationary Borel probability measure on  $X$  and  $\tau$  is the map given by (5.5). We also suppose that  $G$  is not compact (the very easy case in which  $G$  is compact is discussed in Lemma 8.4).

The proof, which we will give from here to the end of the paper, relies on the properties of the dynamical systems

$$(B^X, \mathcal{B}^X, \beta^X, T^X) \text{ and } (B^{\tau, X}, \mathcal{B}^{\tau, X}, \beta^{\tau, X}, T^{\tau, X})$$

which we introduced in sections §3.1 and §3.3, for these values of  $G, V, X, \tau, \dots$

**6.2. Recurrence off the diagonal.** *We now verify condition (HC) of §3.4, which will allow us to apply Proposition 3.9.*

For any  $x \in X$ , denote by  $r_x$  the radius of injectivity at  $x$ , that is the least upper bound of  $r > 0$  such that the map  $V \rightarrow X$ ,  $w \mapsto e^w x$  is injective on the ball  $B(0, r)$ .

**Proposition 6.1.** *In the two cases of §6.1, the averaging operator  $A_\mu$  on  $X \times X$  satisfies condition (HC).*

The proof of this proposition uses ideas of Eskin and Margulis [6]. We note the contrast between Proposition 6.1 and Theorem 1 of LePage in [14], who shows that on the flag variety, a positive power of the distance is contracted under convolution. We will need the following two lemmas. We will use the same notation  $A_\mu$  to denote all the averaging operators of  $\mu$  on every space on which  $\Gamma_\mu$  acts. The first lemma, due to Eskin and Margulis, exhibits a function on which  $A_\mu$  acts by contraction.

**Lemma 6.2** ([6]). *Let  $V = \mathbb{R}^d$  and let  $G$  be a semi-simple Lie subgroup of  $\mathrm{GL}(V)$  such that, for any nonzero  $G$ -invariant subspace  $V' \subset V$ , the image of  $G$  in  $\mathrm{GL}(V')$  is not compact. Denote by  $\varphi$  the function  $\varphi : V \setminus \{0\} \rightarrow \mathbb{R}^*$ ,  $v \mapsto \|v\|^{-1}$ . Then there is  $a_0 < 1$ ,  $\delta_0 > 0$  and  $n_0 \geq 1$ , such that*

$$A_\mu^n(\varphi^\delta) \leq a_0^n \varphi^\delta, \text{ for any } \delta \leq \delta_0 \text{ and } n \geq n_0. \quad (6.1)$$

*Proof.* This is Lemma 4.2 of [6]. It is proved by developing the second order term of  $e^{-\delta \log(\|gv\|/\|v\|)}$  and using the theorem of Furstenberg and Kesten on the positivity of the first Lyapunov exponent  $\lambda_1$ .  $\square$

Whenever  $X$  is noncompact, we will need a variant of a Lemma of Eskin and Margulis which shows the existence of a proper function on  $X$  which is contracted, with a fixed constant, by the averaging operator.

**Lemma 6.3.** *Let  $G$  be a real semisimple connected Lie group without compact factors, let  $\Lambda$  be a lattice in  $G$ , let  $X = G/\Lambda$ , and let  $\mu$  be a compactly supported probability measure on  $G$  whose support generates a Zariski-dense semigroup. Then there is a proper function  $u : X \rightarrow [0, \infty)$  and constants  $a < 1$ ,  $C > 0$  and  $\kappa > 0$ , such that*

$$A_\mu(u) \leq au + C \quad (6.2)$$

and, for every  $x \in X$ ,

$$u(x) \geq r_x^{-\kappa}. \quad (6.3)$$

*Proof.* Since the center of  $G$  intersects  $\Lambda$  in a finite-index subgroup, we may assume with no loss of generality that  $G$  is adjoint and hence linear. In §3.2 of [6], a proper function  $u$  satisfying (6.2) is constructed

explicitly. Due to this construction, if we regard  $G$  as a group of matrices, there exist constants  $C_0 > 0$  and  $\kappa_0 > 0$  such that, for every  $x = g\Lambda \in X$ , we have the lower bound

$$u(x) \geq C_0 \min_{\gamma \in \Lambda} \|g\gamma\|^{\kappa_0}. \quad (6.4)$$

Therefore it suffices to note that there exist constants  $C_1 > 0$  and  $\kappa_1 > 0$  such that, for every  $x = g\Lambda \in X$ , we have the lower bound

$$r_x \geq C_1 \left( \min_{\gamma \in \Lambda} \|g\gamma\| \right)^{-\kappa_1}. \quad (6.5)$$

In fact, if  $h = e^w$  is a nontrivial element of  $G$  with  $hx = x$ , then for any  $\gamma \in \Lambda$ ,  $\delta = \gamma^{-1}g^{-1}hg\gamma$  is in  $\Lambda$  and we have

$$\|h - e\| \geq \|\delta - e\| \|\text{Ad}(g\gamma)^{-1}\|^{-1}.$$

Thus, when denoting  $C_2 = \min_{\delta \in \Lambda \setminus \{e\}} \|\delta - e\|$ , we have

$$\min_{hx=x, h \neq e} \|h - e\| \geq C_2 \left( \min_{\gamma \in \Lambda} \|\text{Ad}(g\gamma)^{-1}\| \right)^{-1}.$$

The lower bound (6.5) follows.  $\square$

*Proof of Proposition 6.1.* First we remark that if the condition (HC) is satisfied for some power  $\mu^{*n_0}$ , then it is satisfied for  $\mu$ . We choose  $a_0 < 1$ ,  $\delta \in (0, 1)$  and  $n_0 \geq 1$  as in Lemma 6.2. By replacing  $\mu$  with  $\mu^{*n_0}$ , we can assume that  $n_0 = 1$ . Let  $\delta \leq \delta_0$ .

For any  $x \neq x'$  in  $X$ , we denote by  $r_{x,x'} = \frac{1}{2} \min(r_x, r_{x'})$ ,

$$d_0(x, x') = \begin{cases} \|w\| & \text{if } x' = e^w x \text{ with } w \in V, \|w\| \leq r_{x,x'}, \\ r_{x,x'} & \text{otherwise} \end{cases},$$

$$v_0(x, x') = d_0(x, x')^{-\delta}.$$

Whenever  $X$  is compact, the function  $v = v_0$  can be used. In the general case, we introduce the function  $u$  and constant  $a < 1, C > 0$  and  $\kappa > 0$  given by Lemma 6.3. We may suppose that  $a = a_0$ . We set  $R_0 = \sup_{g \in \text{supp } \mu} \max(\|R(g)\|, \|R(g)^{-1}\|)$ . If one chooses  $\delta < \kappa$  and  $C_0 = \frac{2R_0^{2\delta}}{1-a_0}$ , then the function  $v$ , given for any  $x \neq x'$  in  $X$  by

$$v(x, x') = v_0(x, x') + C_0(u(x) + u(x')), \quad (6.6)$$

satisfies condition (HC).

In fact, if  $d_0(x, x') \geq R_0^{-1} r_{x,x'}$  then by (6.3),

$$\begin{aligned} (A_\mu v_0)(x, x') &\leq R_0^{2\delta} r_{x,x'}^{-\delta} \\ &\leq 2R_0^{2\delta} (r_x^{-\delta} + r_{x'}^{-\delta}) \leq 2R_0^{2\delta} (u(x) + u(x')). \end{aligned}$$



On the other hand, if  $d_0(x, x') \leq R_0^{-1} r_{x, x'}$ , then, when writing  $x' - e^w x$  with  $w \in V$ ,  $\|w\| \leq r_{x, x'}$ , we have, for any  $g \in G$  of norm at most  $R_0$ ,

$$v_0(gx, gx') = \|gw\|^{-\delta},$$

and hence, by (6.1),

$$(A_\mu v_0)(x, x') \leq a_0 \|w\|^{-\delta} = a_0 v_0(x, x').$$

In both cases, we have therefore the upper bound

$$(A_\mu v_0)(x, x') \leq a_0 v_0(x, x') + R_0^{2\delta} (u(x) + u(x')).$$

Inequality (6.2) and the definition (6.6) of  $v$  thus give the upper bound

$$\begin{aligned} (A_\mu v)(x, x') &\leq a_0 v_0(x, x') + (R_0^{2\delta} + a_0 C_0)(u(x) + u(x')) + 2CC_0 \\ &\leq \frac{1 + a_0}{2} v(x, x') + 2CC_0, \end{aligned}$$

which yields property (HC).  $\square$

**6.3. Recurrence off of finite orbits.** *In this section we exhibit the phenomenon of recurrence away from finite orbits for random walks on  $X$ , analogous to the phenomenon of recurrence to compact subsets in [6].*

**Proposition 6.4.** *In the two cases of §6.1, let  $F$  be a finite  $\Gamma$ -invariant set. Then for any  $\varepsilon > 0$ , there is a compact subset  $K_\varepsilon$  of  $F^c$  such that for any  $x \in X \setminus F$ , there is a constant  $M = M_x$ , which can be chosen to be uniform for  $x$  in a compact subset of  $X \setminus F$ , such that for all  $n \geq M$ ,*

$$A_\mu^n(1_{K_\varepsilon}) \geq 1 - \varepsilon.$$

We will need the following two lemmas.

The first translates the phenomenon of recurrence to compact subsets, due to Foster, and utilized in this context by Eskin and Margulis.

**Lemma 6.5** ([6]). *Let  $H$  be a locally compact group acting continuously on a locally compact space  $Y$ , and let  $\mu$  be a Borel probability measure on  $H$ .*

*Suppose that there is a proper map  $f : Y \rightarrow [0, \infty)$ , and constants  $a < 1, b > 0$  such that  $A_\mu(f) \leq af + b$ .*

*Then for any  $\varepsilon > 0$  there is a compact  $K \subset Y$  such that for every  $y \in Y$ , there is a constant  $M_y$ , which can be chosen to be uniform in  $y$  for  $y$  in a compact subset of  $Y$ , such that for all  $n \geq M$ ,*

$$A_\mu^n(1_K) \geq 1 - \varepsilon.$$

We recall the short proof of this lemma.

*Proof.* By hypothesis, we have for each  $n \geq 1$ ,

$$A_\mu^n(f) \leq a^n f + b(1 + \cdots + a^{n-1}) \leq a^n f + B,$$

where  $B = \frac{b}{1-a}$ . Since  $f$  is proper, we can choose as our compact subset

$$K = \left\{ z \in Y : f(z) \leq \frac{2B}{\varepsilon} \right\}$$

which implies that  $1^{K^c} \leq \frac{\varepsilon}{2B} f$ . Therefore we have the upper bounds

$$A_\mu^n(1_{K^c}) \leq \frac{\varepsilon}{2B} A_\mu^n(f)(y) \leq \frac{\varepsilon a^n}{2B} f(y) + \frac{\varepsilon}{2} \leq \varepsilon,$$

whenever  $n$  is sufficiently large so that  $f(y) \leq \frac{B}{a^n}$ .  $\square$

The second Lemma is a variant of Proposition 6.1.

**Lemma 6.6.** *In the two cases of §6.1, let  $F \subset X$  be a finite  $\Gamma$ -invariant subset. Then there is a proper map  $u_F : X \setminus F \rightarrow [0, \infty)$  and constants  $a < 1, C > 0$  such that*

$$A_\mu(u_F) \leq a u_F + C. \quad (6.7)$$

*Proof.* We proceed as in the proof of Proposition 6.1. We choose  $a_0 < 1$ ,  $\delta_0 > 0$  and  $n_0 \geq 1$  as in Lemma 6.2. By replacing  $\mu$  with  $\mu^{*n_0}$  if necessary, we may assume that  $n_0 = 1$ . Let  $\delta \leq \delta_0$ .

Let  $r_0 > 0$  be a real number such that for every  $x_0 \in F$ , there is  $r_0 \leq \frac{1}{2}r_{x_0}$  such that for every pair  $x_0, x'_0$  of distinct points of  $F$ ,  $r_0 \leq \frac{1}{2}d_0(x_0, x'_0)$ . For any  $x \in X$ , we denote

$$d_0(x) = \begin{cases} \|w\| & \text{if } x = e^w x_0 \text{ with } x_0 \in F \text{ and } \|w\| \leq r_0 \\ r_0 & \text{otherwise} \end{cases}$$

and

$$u_0(x) = d_0(x)^{-\delta}.$$

Whenever  $X$  is compact, the function  $u_F = u_0$  satisfies the requirements. In the general case, the function  $u_F = u_0 + u$  as in Lemma 6.3 satisfies the requirement. The presence of  $u$  is needed only to assure the property of  $u_F$ . To check that  $u_F$  satisfies the requirements, we set

$$R_0 = \sup_{g \in \text{supp } \mu} \max(\|R(g)\|, \|R(g)^{-1}\|).$$

On one hand, if  $d_0(x) \geq R_0^{-1}r_0$  then we have

$$(A_\mu u_0) \leq R_0^{2\delta} r_0^{-\delta}.$$

On the other hand, if  $d_0(x) \leq R_0^{-1}r_0$  then, when writing  $x = e^w x_0$  with  $x_0 \in F$ , we have for every  $g \in G$  of norm at least  $R_0$ ,

$$d_0(gx) \leq \|gq\|^{-\delta},$$

and thus, by (6.1),

$$(A_\mu u_0)(x) \leq \|w\|^{-\delta} = a_0 u_0(x).$$

In all cases, we have the upper bound

$$(A_\mu u_0)(x) \leq a_0 u_0(x) + R_0^{2\delta} r_0^{-\delta}.$$

This inequality and that of Lemma 6.3 provide the sought-for inequality concerning  $u_F$ .  $\square$

*Proof of Proposition 6.4.* This follows from Lemma 6.5 applied to  $Y = X \setminus F$  and to the function  $f = u_F$  of Lemma 6.6.  $\square$

**6.4. Stationary probability measures on  $G/H$ .** *In order to exploit the drift argument, we will need, in the first case of §6.1, the following proposition which is of independent interest.*

**Proposition 6.7.** *Let  $G$  be a connected semi-simple real Lie group without compact factors,  $\mu$  a compactly supported probability measure whose support generates a Zariski dense subsemigroup in  $G$ , and  $H \subset G$  a unimodular subgroup. If there exists a  $\mu$ -stationary probability measure on the homogeneous space  $G/H$ , then the Lie algebra of  $H$  is an ideal in the Lie algebra of  $G$ .*

For the proof, we will use the following lemma.

**Lemma 6.8.** *Let  $V = \mathbb{R}^d$ , let  $G$  be a semi-simple subgroup of  $\mathrm{GL}(V)$  with no compact factors, and let  $\mu$  be a compactly supported Borel probability measure on  $G$  generating a Zariski dense subsemigroup. Then any  $\mu$ -stationary probability measure  $\nu$  on  $V$  is supported on the subspace  $V^G$  of  $G$ -fixed points in  $V$ .*

*Proof.* Suppose by contradiction that there is a  $\mu$ -stationary probability measure  $\nu$  on  $V$  which is not supported on  $V^G$ . Then there is an irreducible sub-representation  $W \subset V$  of dimension at least 2 such that the projection of  $\nu$  on  $W$  is not a Dirac mass at 0. This projection is also  $\mu$ -stationary. Thus we may assume that  $V$  is irreducible and  $G$  is not compact.

We will use again the Bernoulli system  $(B, \mathcal{B}, \beta, T)$  with alphabet  $(G, \mu)$  and the fibered dynamical system  $B \times V$  equipped with the transformation  $\mathbf{R} : (b, v) \mapsto (Tb, b_0 v)$  which leaves the probability measure  $\beta \otimes \nu$  invariant.

The theorem of Furstenberg and Kesten about the positivity of the first Lyapunov exponent ([10], see also [7], chapter 1) ensures that for  $\beta$ -a.e.  $b \in B$ , there is a subspace  $W_b \subsetneq V$  such that, for any

$v \in V \setminus W_b$ , the norm  $\|b_n \cdots b_0 v\|$  converges (exponentially fast) to infinity. We introduce the  $\mathbf{R}$ -invariant set

$$Z = \{(b, v) \in B \times V : v \notin W_b\}$$

and the function  $\varphi$  on  $Z$  given by

$$\varphi(b, v) = \|v\|.$$

Since  $\nu$  is  $\mu$ -stationary, and since  $\mu$  is Zariski dense in  $G$  and the action of  $G$  on  $V$  is irreducible,  $\nu$  does not give positive mass to any proper subspaces of  $V$ . We therefore have  $(\beta \otimes \nu)(Z) = 1$ . By construction, for  $\beta \otimes \nu$ -a.e.  $z \in Z$ , we have

$$\lim_{n \rightarrow \infty} \varphi(\mathbf{R}^n z) = \infty,$$

which contradicts the Poincaré recurrence theorem.  $\square$

*Proof of Proposition 6.7.* We denote by  $\nu$  a  $\mu$ -stationary probability measure on  $G/H$ , denote by  $\mathfrak{g}$  the Lie algebra of  $G$ , by  $\mathfrak{h}$  that of  $H$ , set  $r = \dim \mathfrak{h}$ ,  $V = S^2(\bigwedge^r \mathfrak{g})$  and let  $v$  be a nonzero point of the line  $S^2(\bigwedge^r \mathfrak{h}) \subset V$ .

Since  $H$  is unimodular,  $H$  is contained in the stabilizer  $N$  of the point  $v$ . Therefore the orbit  $Gv \cong G/N$  also admits a stationary measure: the image  $\nu'$  of  $\nu$  under the projection  $G/H \rightarrow G/N$ . By Lemma 6.8,  $\nu'$  is supported on the subspace  $V^G$  of  $G$ -fixed vectors. Thus  $N = G$ . Since  $N$  normalizes  $\mathfrak{h}$ ,  $\mathfrak{h}$  is an ideal of  $\mathfrak{g}$ .  $\square$

**6.5. Horocycle flows.** *The goal of this section is to construct an action of  $V_0$  which plays a role analogous to the one played by the horocycle flow on compact hyperbolic surfaces, in the sense that the orbits of this action are contained in the stable leaves relative to the factor map  $B^{\tau, X} \rightarrow B^\tau$  and they are uniformly dilated by the semi-flow  $T^\tau$ .*

We keep the notations of §6.1.

**Definition 6.9.** *The horocycle flow is the action  $\Phi$  of  $V_0$  on  $B^{\tau, X}$  given by, for any  $v \in V_0$  and  $\beta^\tau$ -a.e.  $c = (b, k, m) \in B^\tau$  and every  $x \in X$ ,*

$$\Phi_v(c, x) = (c, \exp(D_c(v))x), \quad (6.8)$$

where  $D_c(v)$  is the element of  $V_c$  given by

$$D_c(v) = e^{k - \varphi(b)} s(\xi(b)) m v. \quad (6.9)$$

Recall that  $s, \xi, \varphi$  were defined in §5.1 and §5.2. Geometrically, the flow  $\Phi$  ‘translates every point  $(c, x)$  in the direction of  $V_c$ ’. We note that at this stage in the argument, we do not know that this flow preserves the probability measure  $\beta^{\tau, X}$ : we will know this after having proved

Theorem 1.1. This difficulty is certainly a source of complications which are the heart of the matter.

The fundamental property of the horocycle flow is its relationship with the flow  $(T_\ell^{\tau, X})_{\ell \geq 0}$  on  $B^{\tau, X}$ .

**Lemma 6.10.** *In the two cases of §6.1, for any  $v \in V_0$  and any  $\ell \geq 0$ , we have, for  $\beta^\tau$ -a.e.  $c \in B^\tau$  and any  $x \in X$ ,*

$$T_\ell^{\tau, X} \circ \Phi_v(c, x) = \Phi_{e^{-\ell}v} \circ T_\ell^{\tau, X}(c, x). \quad (6.10)$$

*Proof.* Denote by  $S$  the transformation of  $B \times \mathbb{R} \times M \times X$  given by

$$S(b, k, m, x) = (Tb, k - \tau_{\mathbb{R}}(b), \tau_M(b)m, b_0^{-1}x).$$

We note that  $B^{\tau, X}$  is the set of points in  $B \times \mathbb{R}_+ \times M \times X$  which are taken by  $S$  to points outside of this product.

Introduce the flow  $\tilde{T}_\ell^{\tau, X}$  defined on  $B \times \mathbb{R} \times M \times X$  by

$$\tilde{T}_\ell^{\tau, X}(b, k, m, x) = (b, k + \ell, m, x).$$

The flow  $T_\ell^{\tau, X}$  is given, for  $\ell \geq 0$  and  $(b, k, m, x) \in B^{\tau, X}$ , by

$$T_\ell^{\tau, X}(b, k, m, x) = (S^p \circ \tilde{T}_\ell^{\tau, X})(b, k, m, x)$$

where  $p \geq 0$  is the unique integer for which this expression is in  $B^{\tau, X}$ . We then define an action  $\tilde{\Phi}$  of  $V_0$  on  $B \times \mathbb{R} \times M \times X$  by the formula:

$$\tilde{\Phi}_v(b, k, m, x) = (b, k, m, \exp(D_{(b, k, m)}(v))x)$$

where

$$D_{(b, k, m)}(v) = e^{k - \varphi(b)} s(\xi(b))mv. \quad (6.11)$$

Before continuing we prove the following equality: for  $\beta$ -a.e.  $b \in B$ , every  $(k, m) \in \mathbb{R} \times M$ , and every  $v \in V_0$ , we have

$$b_0^{-1} D_{(b, k, m)}(v) = D_{S(b, k, m)}(v) \quad (6.12)$$

where

$$S(b, k, m) = (Tb, k - \tau_{\mathbb{R}}(b), \tau_M(b)m). \quad (6.13)$$

To this end, we compute as in Lemma 5.4,

$$\begin{aligned} b_0^{-1} D_{(b, k, m)}(v) &= e^{k - \varphi(b)} b_0^{-1} s(\xi(b))mv \\ &= e^{k - \varphi(b)} s(\xi(Tb))\theta(b)^{-1}mv \\ &= e^{k - \varphi(b) - \theta_{\mathbb{R}}(b)} s(\xi(Tb))\theta_M(b)mv, \end{aligned}$$

and hence, using (5.4),

$$\begin{aligned} b_0^{-1} D_{(b, k, m)}(v) &= e^{k - \tau_{\mathbb{R}}(b) - \varphi(Tb)} s(\xi(Tb))\tau_M(b)mv \\ &= D_{S(b, k, m)}(v). \end{aligned}$$

We deduce, thanks to (6.12), the following two equalities

$$S \circ \tilde{\Phi}_v = \tilde{\Phi}_v \circ S \quad (6.14)$$

and

$$\tilde{T}_\ell^{\tau, X} \circ \tilde{\Phi}_v = \tilde{\Phi}_{e^{-\ell}v} \circ \tilde{T}_\ell^{\tau, X} \quad (6.15)$$

which proves that the flow  $\Phi_v$  satisfies (6.10).  $\square$

**6.6. Horocyclic conditional probabilities.** *In this section we introduce the ‘horocyclic conditional function’ and prove that this function is measurable for the tail  $\sigma$ -algebra.*

We keep the notations of §6.1 and also denote by  $t_v$  the translation of  $V_0$  by an element  $v \in V_0$ . We write  $\sigma : B^{\tau, X} \rightarrow \mathcal{M}_1(V_0)$  the map given by ‘conditional measures of the probability measure  $\beta^{\tau, X}$  with respect to the horocyclic action of  $V_0$ ’.

**Lemma 6.11.** *In the two cases of §6.1, there is a Borel subset  $E \subset B^{\tau, X}$  such that  $\beta^{\tau, X}(E^c) = 0$  and such that, for any  $v \in V_0$  and  $(c, x) \in E$  for which  $\Phi_v(c, x) \in E$ , we have*

$$t_{v*}\sigma(\Phi_v(c, x)) \simeq \sigma(c, x). \quad (6.16)$$

*Proof.* This follows from Proposition 4.2.  $\square$

Recall that the symbol  $\simeq$  refers to equality after a normalization by a scalar.

Geometrically, for  $\beta^{\tau, X}$ -a.e.  $(c, x) \in B^{\tau, X}$ ,  $\sigma(c, x)$  is the conditional measure of  $\delta_c \otimes \nu_c$  for the action of  $V_0$  on  $\{c\} \times X$ .

**Lemma 6.12.** *In the two cases of §6.1, for any  $\ell \geq 0$ , for  $\beta^{\tau, X}$ -a.e.  $(c, x) \in B^{\tau, X}$ , we have*

$$\sigma(T_\ell^{\tau, X}(c, x)) \simeq (e^{-\ell})_*\sigma(c, x).$$

In this equality,  $e^{-\ell}$  denotes the homothety by a factor of  $e^{-\ell}$  of  $V_0$ .

*Proof.* This is a result of the uniqueness of  $\sigma$ , equality (6.10) and the fact that for  $\beta$ -a.e.  $b \in B$ , for any  $p \in \mathbb{N}$ , the action of  $b_{p-1}^{-1} \cdots b_0^{-1}$  induces an isomorphism between the measure spaces  $(X, \nu_b)$  and  $(X, \nu_{T^p b})$ .  $\square$

**Corollary 6.13.** *In the two cases of §6.1, the map  $\sigma : B^{\tau, X} \rightarrow \mathcal{M}_1(V_0)$  is  $\mathcal{Q}_\infty^{\tau, X}$ -measurable.*

*Proof.* It suffices to show that for any  $\ell \geq 0$ , it is  $\mathcal{Q}_\ell^{\tau, X}$ -measurable. This results from the equality, for  $\beta^{\tau, X}$ -a.e.  $(c, x) \in B^{\tau, X}$ ,  $\sigma(c, x) \simeq (e^\ell)_*(\sigma(T_\ell^{\tau, X}(c, x)))$ .  $\square$

**6.7. Approach outside the  $W$ -leaves.** *In order to start the drift argument, we need to ensure, in any compact subset of positive  $\beta^{\tau,X}$ -measure, that a.e. point  $x$  is approached by points which are not in the same leaf as  $x$  for a certain subfoliation of the relative stable leaf.*

For  $b \in B$ , we introduce a vector subspace of  $V$ :

$$W_b = \left\{ v \in V : \sup_{p \in \mathbb{N}} (e^{\theta_{\mathbb{R},p}(b)} \|b_p^{-1} \cdots b_0^{-1} v\|) < \infty \right\} \quad (6.17)$$

and, for  $c = (b, k, m) \in B^\tau$ , we set  $W_c = W_b$ .

**Lemma 6.14.** *In the two cases of §6.1, for  $\beta^X$ -a.e.  $(b, x)$  in  $B^X$ , we have  $\nu_b(\exp(W_b)x) = 0$ .*

*Proof.* By ergodicity of the Bernoulli system  $(B, \mathcal{B}, \beta, T)$  and by Furstenberg's formula (5.3), for  $\beta$ -a.e.  $b \in B$  we have  $\lim_{p \rightarrow \infty} \frac{1}{p} \theta_{\mathbb{R},p}(b) = \int_B \theta_{\mathbb{R}}(b) d\beta(b) = \lambda_1 > 0$ . Therefore, by Lemma 5.4, for every  $v \in W_b$ , we have  $\lim_{p \rightarrow \infty} \|b_p^{-1} \cdots b_0^{-1} v\| = 0$ . Choosing a distance function  $d$  on  $X$ , gives a right-invariant distance on the group  $\tilde{X}$ , the universal cover of  $X$ . For  $\beta$ -a.e.  $b \in B$ , every  $x \in X$ , and every  $v \in W_b$ , we have

$$d(b_p^{-1} \cdots b_0^{-1} \exp(v)x, b_p^{-1} \cdots b_0^{-1} x) \rightarrow_{p \rightarrow \infty} 0.$$

By Proposition 6.1, the measure  $\mu$  satisfies property (HC), and hence, by Proposition 3.9, for  $\beta^X$ -a.e.  $(b, x) \in B^X$ , we have  $\nu_b(\exp(W_b)x) = 0$ , as required.  $\square$

**Corollary 6.15.** *In the two cases of §6.1, let  $F \subset B^{\tau,X}$  be a  $\mathcal{B}^{\tau,X}$ -measurable subset such that  $\beta^{\tau,X}(F) > 0$ . Then, for  $\beta^{\tau,X}$ -a.e.  $(c, x) \in F$ , there is a sequence  $(u_n)$  of elements of  $V \setminus W_c$  such that  $u_n \rightarrow 0$  and such that, for every  $n$ ,  $(c, \exp(u_n)x) \in F$ .*

*Proof.* Let  $(U_n)$  be a countable basis of neighborhoods of 0 in  $V$ . For  $\beta^\tau$ -a.e.  $c \in B^\tau$ , the set  $F_c = \{x \in X : (c, x) \in F\}$  satisfies  $\nu_c(F_c) > 0$ . For  $\beta^{\tau,X}$ -a.e.  $(c, x) \in F$ , for every  $n \geq 0$  we therefore have  $\nu_c(F_c \cap \exp(U_n)x) > 0$  and since, by Lemma 6.14,  $\nu_c(\exp(W_c)x) = 0$ , we have  $\nu_c(F_c \cap (\exp(U_n) \setminus W_c)x) > 0$ .  $\square$

## 7. INVARIANCE OF STATIONARY MEASURES

*The goal of this chapter is to present the exponential drift argument and to deduce invariance properties for certain conditional measures of stationary measures (Proposition 7.6).*

*To this end we collect the pieces of the puzzle which we have prepared in previous chapters.*

**7.1. The exponential drift.** *The heart of this paper is the following proposition.*

We keep as always the notation of §6.1. In particular,  $\mu$  is a probability measure on  $G$  whose support generates a Zariski-dense subsemigroup,  $\nu$  is a  $\mu$ -stationary and  $\mu$ -ergodic Borel probability measure on  $X$ , and the symbols  $s, \xi, \theta, \theta_{\mathbb{R}}, \varphi, \tau_{\mathbb{R}}, \tau_M, \tau, B^\tau, \beta^\tau, \beta^{\tau, X}, \sigma, R$ , etc., have the same meanings as in §5 and §6.

**Proposition 7.1.** *In the two cases of §6.1, let  $(Y, \mathcal{Y})$  be a standard Borel space, let  $f : B^{\tau, X} \rightarrow Y$  be a  $\mathcal{Q}_\infty^{\tau, X}$ -measurable map, and let  $E \subset B^{\tau, X}$  be a  $\mathcal{B}^{\tau, X}$ -measurable subset such that  $\beta^{\tau, X}(E^c) = 0$ . Then for  $\beta^{\tau, X}$ -a.e.  $(c, x) \in B^{\tau, X}$ , for any  $\varepsilon > 0$ , there exists a nonzero element  $v \in V_0$  of norm at most  $\varepsilon$  and an element  $(c', x') \in E$  such that  $\Phi_v(c', x')$  is also in  $E$  and such that*

$$f(\Phi_v(c', x')) = f(c', x') = f(c, x). \quad (7.1)$$

**Remark 7.2.** *Since we do not yet know that the horocycle flow preserves the measure  $\beta^{\tau, X}$  (we will show this in §8.1), it is not a priori clear that there exists an element  $(c', x') \in E$  and a nonzero vector  $v \in V_0$  such that  $\Phi_v(c', x')$  is in  $E$ . This assertion will be a nontrivial consequence of Proposition 7.1.*

*Beginning of proof of Proposition 7.1.* By definition, we can assume that  $Y$  is endowed with the topology of a complete separable metric space for which  $\mathcal{Y}$  is the Borel  $\sigma$ -algebra. Similarly we can choose the topology of a compact metric space on  $B^\tau$  so that the Borel  $\sigma$ -algebra coincides, up to adding subsets of measure zero, with  $\mathcal{B}^\tau$ , and such that the natural projection  $B^\tau \rightarrow M$  is continuous, and endow  $B^\tau \times X$  with the product topology of this topology and the usual topology on  $X$ .

Let  $\alpha > 0$  be a small number. By Lusin's theorem, there is a compact subset  $K \subset E$  in  $B^{\tau, X}$  such that  $\beta^{\tau, X}(K^c) < \alpha^2$  and such that all the functions we will encounter, such as the functions  $f, \theta, (c, x) \mapsto \varphi(b), (c, x) \mapsto V_c$  and also  $(c, x) \mapsto D_c \in \text{Hom}(V_0, V_c)$ , are uniformly continuous on  $K$ .

The proof relies on the study of the function  $\mathbb{E} \left( 1_K | \mathcal{Q}_\infty^{\tau, X} \right)$ .

On one hand, this function is bounded above by 1 and its average is bounded below by  $1 - \alpha^2$ , because:

$$\int_{B^{\tau, X}} \mathbb{E} \left( 1_K | \mathcal{Q}_\infty^{\tau, X} \right) (c, x) d\beta^{\tau, X}(c, x) = \beta^{\tau, X}(K) > 1 - \alpha^2. \quad (7.2)$$

Thus the function  $\mathbb{E} \left( 1_K | \mathcal{Q}_\infty^{\tau, X} \right)$  is bounded below by  $1 - \alpha$  on a set of measure  $1 - \alpha$ . Therefore there is a compact subset  $L \subset E$  in  $B^{\tau, X}$



such that  $\beta^{\tau,X}(L^c) < \alpha$  and such that, for every  $(c, x) \in L$ , we have

$$\mathbb{E}(1_K | \mathcal{Q}_\infty^{\tau,X})(c, x) > 1 - \alpha. \quad (7.3)$$

By Lusin's theorem, we may also suppose that  $f$  is continuous on  $L$ .

On the other hand, by the Martingale convergence theorem, for  $\beta^{\tau,X}$ -a.e.  $(c, x) \in B^{\tau,X}$ , we have

$$\lim_{\ell \rightarrow \infty} \mathbb{E}(1_K | \mathcal{Q}_\ell^{\tau,X})(c, x) = \mathbb{E}(1_K | \mathcal{Q}_\infty^{\tau,X})(c, x). \quad (7.4)$$

By Corollary 3.8, we may also suppose that for every  $(c, x) \in L$  and  $\ell$  rational, the left hand side of (7.4) is given by formula (3.11). Thanks to the law of the last jump (Proposition 2.3), recalling the notation  $h_{\ell,c}(a)$ , this can be rewritten as

$$\mathbb{E}(1_K | \mathcal{Q}_\ell^{\tau,X})(c, x) = \int_B 1_K(h_{\ell,c,x}(a)) d\beta(a), \quad (7.5)$$

where

$$h_{\ell,c,x}(a) = (c', x') \text{ with } c' = h_{\ell,c}(a) \text{ and } x' = \rho_\ell(c')^{-1} \rho_\ell(c)x.$$

Moreover, since  $f$  is  $\mathcal{Q}_\infty^{\tau,X}$ -measurable, it is  $\mathcal{Q}_\ell^{\tau,X}$ -measurable for each  $\ell \geq 0$ , and hence, again by Corollary 3.8 and Proposition 2.3, we can also assume that for every  $(c, x) \in K$ , for  $\beta$ -a.e.  $a \in B$ , for any rational  $\ell \geq 0$ , we have  $f(h_{\ell,c,x}(a)) = f(c, x)$ .

Egorov's theorem ensures that, outside a subset of  $L$  of arbitrarily small  $\beta^{\tau,X}$ -measure, the convergence in (7.4) is uniform on  $L$ . Therefore, after removing a subset of  $L$  of small measure, there exists  $\ell_0 \geq 0$  such that for every integer  $\ell \geq \ell_0$ , for every  $(c, x) \in L$ , we have

$$\mathbb{E}(1_K | \mathcal{Q}_\ell^{\tau,X})(c, x) \geq 1 - \alpha. \quad (7.6)$$

Since the  $\beta^{\tau,X}$ -measure of  $L^c$  is at most  $\alpha$  and  $\alpha$  was chosen arbitrarily small, it suffices to prove (7.1) for  $\beta^{\tau,X}$ -a.e.  $(c, x) \in L$ .

By Corollary 6.15 we may suppose that for the points  $(c, x) \in L$ , there exists a sequence  $(u_n)$  of elements of  $V \setminus W_c$  which converge to 0 and such that the points  $(c, y_n)$  defined by  $(c, y_n) = (c, \exp(u_n)x)$  are also in  $L$ .

We apply the two formulas (7.5) and (7.6) to the conditional expectations at the two points  $(c, x)$  and  $(c, y_n)$ . For  $\ell \geq \ell_0$ , we then have

$$\beta\{a \in B : h_{\ell,c,x}(a) \in K\} \geq 1 - \alpha \quad (7.7)$$

and

$$\beta\{a \in B : h_{\ell,c,y_n}(a) \in K\} \geq 1 - \alpha. \quad (7.8)$$

We will now say a few words about the strategy of proof. By construction, for  $y = \exp(u)x$  with  $u \in \mathfrak{g}$ , the parameterizations of the two fibers of  $T_\ell^{\tau, X}$  passing through  $(c, x)$  and  $(c, y)$  are related by a *drift* that can be easily computed: if  $(c', x') = h_{\ell, c, x}(a)$  and  $(c', y') = h_{\ell, c, y}(a)$ , then we have

$$y' = \exp(F_{\ell, c}(a)u)x' \quad (7.9)$$

where the drift is given by

$$F_{\ell, c}(a)u = R_\ell(c') \circ R_\ell(c)^{-1}(u), \quad (7.10)$$

and where, as in §3.3, if we write  $c = (b, k, m)$  and  $p = p_\ell(b, k)$  then we have

$$R_\ell(c) = R(b_0) \circ \cdots \circ R(b_{p-1}). \quad (7.11)$$

To simplify the notations, we will sometimes write  $b_0$  for  $R(b_0)$ . We will see that, for the parameterization of the two fibers of  $T_\ell^{\tau, X}$  passing through the points  $(c, x)$  and  $(c, y_n)$ , a large proportion of the parameters  $a \in B$  correspond to two points  $(c'_n, x'_n)$  and  $(c'_n, y'_n)$  which are both in  $K$ . We will now adjust the line  $\ell = \ell_n$  of the sequence  $u_n$  in order to control the norm and the direction of the drift separating these two points.

This will be possible thanks to the following lemma.

**Lemma 7.3.** *In the two cases of §6.1, for any  $\alpha > 0$  and  $\eta > 0$ , there is  $r_0 \geq 1$ , such that for  $\beta^\tau$ -a.e.  $c \in B^\tau$ , for all  $\ell$  sufficiently large, we have for all  $u \in V \setminus \{0\}$ ,*

$$\beta \left\{ a \in B : \frac{1}{r_0} \leq \frac{\|F_{\ell, c}(a)u\|}{e^{\theta_{\mathbb{R}, \ell}(c)} \|R_\ell(c)^{-1}u\|} \leq r_0 \right\} \geq 1 - \alpha. \quad (7.12)$$

and

$$\beta \left\{ a \in B : d(\mathbb{R}F_{\ell, c}(a)u, \mathbb{P}(V_{h_{\ell, c}(a)})) \leq \eta \right\} \geq 1 - \alpha. \quad (7.13)$$

*Proof.* Recall that by §2.3, for  $\beta^\tau$ -a.e.  $c \in B^\tau$ , for  $\beta$ -a.e.  $a \in B$ , we have  $\lim_{\ell \rightarrow \infty} q_{\ell, c}(a) = \infty$ .

In order to obtain the upper bound (7.12), we apply Corollary 5.5(a) with the vectors  $v_1 = R_\ell(c)^{-1}u$  and  $v_2 \in V_{T_\ell^\tau(c)}$  which results in the equality

$$\frac{\|F_{\ell, c}(a)u\|}{\|R_\ell(c)^{-1}u\|} = \frac{\|a_{q-1} \cdots a_0 v_1\|}{\|v_1\|}$$

for  $q = q_{\ell, c}(a)$ , and, thanks to Lemma 5.4, in the equality

$$e^{\theta_{\mathbb{R}, \ell}(c')} = \frac{\|a_{q-1} \cdots a_0 v_2\|}{\|v_2\|}$$

with  $c' = h_{\ell,c}(a)$ . In order to obtain (7.13), we apply Corollary 5.5(b) with the same vector  $v = R_\ell(c)^{-1}u$  and with  $W = V_{T_\ell^\tau(c)}$ . For  $\beta^\tau$ -a.e.  $c \in B^\tau$ , for every  $\alpha, \eta > 0$  there is thus  $\ell_0 \geq 0$  such that for every  $u \in V \setminus \{0\}$  and  $\ell \geq \ell_0$ ,

$$\beta \left\{ a \in B : d(\mathbb{R}F_{\ell,c}(a)u, \mathbb{P}(V_{h_{\ell,c}(a)})) \leq \eta \right\} \geq 1 - \alpha,$$

as required.  $\square$

*End of proof of Proposition 7.1.* We now explain our strategy in more detail. We will choose the parameter  $\ell = \ell_n$  in the following manner.

Since the measure  $\mu$  on  $G$  is compactly supported, and since the section  $s$  in §5.2 has a bounded image, there is  $C_0 > 0$  such that, for  $\beta$ -a.e.  $b \in B$ , for every  $u \in V \setminus \{0\}$ , and every  $p \in \mathbb{N}$ , we have

$$\frac{e^{\theta_{\mathbb{R},p+1}(b)} \|b_{p+1}^{-1} \cdots b_0^{-1} u\|}{e^{\theta_{\mathbb{R},p}(b)} \|b_p^{-1} \cdots b_0^{-1} u\|} \leq C_0.$$

Since  $u_n$  is not in  $W_c$ , the sequence  $p \mapsto e^{\theta_{\mathbb{R},p}(b)} \|b_p^{-1} \cdots b_0^{-1} u_n\|$  is not bounded above. For  $n$  large enough, there is therefore an integer  $p_n$  such that

$$\frac{e^{-M_0\varepsilon}}{r_0 C_0} \leq e^{\theta_{\mathbb{R},p_n}(b)} \|b_{p_n}^{-1} \cdots b_0^{-1} u_n\| \leq \frac{e^{-M_0\varepsilon}}{r_0} \quad (7.14)$$

where  $M_0 = \sup \tau$ . We choose a rational  $\ell_n$  such that  $p_n = p_{\ell_n}(c)$ . This is possible since  $\tau$  is strictly positive.

Hence, since  $\alpha < \frac{1}{4}$ , we can choose an element  $a = a_n \in B$  such that it simultaneously belongs to the sets given by (7.7) and (7.8), (7.12) and (7.13) with  $\ell = \ell_n$ ,  $u = u_n$  and  $\eta = \eta_n \rightarrow 0$  and such that

$$f(h_{\ell_n,c,x}(a_n)) = f(c, x) \text{ and } f(h_{\ell_n,c,y_n}(a_n)) = f(c, y_n). \quad (7.15)$$

Up to passing to a subsequence, we have

- (1) The sequence  $(c'_n, x'_n) = h_{\ell_n,c,x}(a_n)$  has a limit  $(c', x') \in K$ ,
- (2) The sequence  $(c'_n, y'_n) = h_{\ell_n,c,y_n}(a_n)$  has a limit in  $K$ , and
- (3) the limit of the drift vector  $w = \lim_{n \rightarrow \infty} F_{\ell_n,c}(a_n)u_n$  exists, is nonzero, is of norm at most  $e^{-M_0\varepsilon}$ , and belongs to  $V_{c'}$ .

We then deduce, by passing to a limit in (7.15), since all the limits considered have their values in  $K$  and  $L$ , and since  $f$  is continuous on these sets,

$$f(c', x') = \lim_{n \rightarrow \infty} f(c'_n, x'_n) = \lim_{n \rightarrow \infty} f(c, x) = f(c, x),$$

$$f(c', y') = \lim_{n \rightarrow \infty} f(c'_n, y'_n) = \lim_{n \rightarrow \infty} f(c, y_n) \text{ and } y' = \exp(w)x'.$$

In addition, if we let  $v \in V_0$  be the nonzero vector  $v = D_{c'}^{-1}(w)$ , we have

$$\|v\| \leq \varepsilon \text{ and } (c', y') = \Phi_v(c', x'),$$

which is the sought-for conclusion.  $\square$

**7.2. Stabilizers of conditional measures.** *We will make explicit the information furnished by the drift argument, regarding the horocyclic conditional measures  $\sigma(c, x)$ , for  $\beta^{\tau, X}$ -a.e.  $(c, x) \in B^{\tau, X}$ .*

We introduce the connected stabilizers of the measures  $\sigma(c, x)$  and their class  $R_+^* \sigma(c, x)$  modulo normalization:

$$J(c, x) = \{v \in V_0 : t_{v*} \sigma(c, x) = \sigma(c, x)\}_0,$$

$$J_1(c, x) = \{v \in V_0 : t_{v*} \sigma(c, x) \simeq \sigma(c, x)\}_0.$$

These are closed subgroups and hence vector subspaces of  $V_0$ .

**Proposition 7.4.** *In the two cases of §6.1, for  $\beta^{\tau, X}$ -a.e.  $(c, x) \in B^{\tau, X}$ , we have*

- a)  $J_1(c, x) \neq \{0\}$ ,
- b)  $J(c, x) = J_1(c, x)$ .

*Proof.* a) We will show, for  $\beta^{\tau, X}$ -a.e.  $(c, x)$  and every  $\varepsilon > 0$ , the stabilizer of  $\sigma(c, x)$  modulo normalization contains a nonzero vector of norm at most  $\varepsilon$ .

By Lemma 6.11, there is a Borel subset  $E \subset B^{\tau, X}$  such that  $\beta^{\tau, X}(E^c) = 0$  and such that, for every  $v \in V_0$  and  $(c', x') \in E$  such that  $\Phi_v(c', x') \in E$ , we have

$$t_{v*} \sigma(\Phi_v(c', x')) \simeq \sigma(c', x'). \quad (7.16)$$

By Corollary 6.13, the function  $\sigma$  is  $\mathcal{Q}_\infty^{\tau, X}$ -measurable. The drift (Proposition 7.1) applied to this set  $E$  and this function  $f = \sigma$  produces, for  $\beta^{\tau, X}$ -a.e.  $(c, x) \in B^{\tau, X}$  and every  $\varepsilon > 0$ , a nonzero vector  $v \in V_0$  of norm at most  $\varepsilon$  and an element  $(c', x')$  of  $E$  such that  $\Phi_v(c', x')$  is also in  $E$  and such that

$$\sigma(\Phi_v(c', x')) \simeq \sigma(c', x') \simeq \sigma(c, x).$$

By applying (7.16) to this element  $(c', x')$ , we find

$$t_{v*} \sigma(\Phi_v(c', x')) \simeq \sigma(c', x')$$

and hence

$$t_{v*} \sigma(c, x) \simeq \sigma(c, x).$$

The vector  $v$  is indeed in the stabilizer of  $\sigma(c, x)$  modulo normalization. The stabilizer is non-discrete and closed. It thus contains a nonzero linear subspace of  $V_0$ .

b) For  $\beta^{\tau, X}$ -a.e.  $(c, x) \in B^{\tau, X}$ , there is a linear form  $\alpha(c, x) \in J_1(c, x)^*$  such that, for any  $v \in J_1(c, x)$ ,

$$t_{v*} \sigma(c, x) = e^{\alpha(c, x)(v)} \sigma(c, x).$$

We wish to show  $\alpha = 0$ . Lemma 6.12 implies, for  $\beta^{\tau, X}$ -a.e.  $(c, x) \in B^{\tau, X}$ , the equality  $J_1(T_\ell^{\tau, X}(c, x)) = J_1(c, x)$  and, for every  $\ell \geq 0$ , the equality of linear forms on  $J_1(c, x)$ :

$$\alpha(T_\ell^{\tau, X}(c, x)) = e^\ell \alpha(c, x), \quad (7.17)$$

from which it follows, after applying the Poincaré recurrence theorem, that  $\beta^{\tau, X}$ -almost surely,  $\alpha = 0$ .  $\square$

**7.3. Disintegration of  $\nu_b$  along the stabilizers.** *In this section we will disintegrate the limit measures  $\nu_b$  along the connected components of the stabilizers of the horocyclic conditional. We will find that the measures  $\nu_{b,x}$  are invariant under a nontrivial unipotent group.*

We will begin by translating the fact that the stabilizers of the conditional horocyclic measures are not discrete into a statement which does not involve the suspension  $B^\tau$ .

For  $\beta$ -a.e.  $b \in B$ , and  $\nu_b$ -a.e.  $x \in X$ , we denote by  $\sigma_{b,x} \in \mathcal{M}(V_b)$  the conditional measure at  $x$  of  $\nu_b$  for the action on  $X$  of  $V_b$  through the group  $\exp(V_b)$  (see §4.1), and we denote  $V_{b,x} \subset V_b$  the connected component of the stabilizer of  $\sigma_{b,x}$  in  $V_b$ .

**Proposition 7.5.** *In the two cases of §6.1, for  $\beta^X$ -a.e.  $(b, x) \in B^X$ , we have  $\sigma_{b,x} \simeq b_{0*}\sigma_{T^X(b,x)}$ ,  $V_{b,x} = b_0(V_{T^X(b,x)})$  and  $V_{b,x} \neq 0$ .*

*Proof.* The first equality follows from the equalities, for  $\beta$ -a.e.  $b \in B$ ,  $\nu_{Tb} = (b_0^{-1})_*\nu_b$  and, for every  $x \in X$  and  $v \in \mathfrak{g}$ ,

$$T^X(b, \exp(v)x) = (Tb, \exp(b_0^{-1}v)b_0^{-1}x).$$

The second equality follows.

The fact that  $V_{b,x}$  is nonzero follows from Proposition 7.4 and the equality, for  $\beta^{\tau, X}$ -a.e.  $(c, x) \in B^{\tau, X}$ ,  $V_{b,x} = R(s(\xi(b))m)(J(c, x))$ , where  $c = (b, k, m)$ .  $\square$

The disintegration of  $\beta^X$  along the map  $(b, x) \mapsto (b, V_{b,x})$ , or, what will turn out to be the same, the disintegration for  $\beta$ -a.e.  $b$  of  $\nu_b$  along the map  $x \mapsto V_{b,x}$ , can be written as

$$\nu_b = \int_X \nu_{b,x} d\nu_b(x)$$

where, for  $\beta^X$ -a.e.  $(b, x) \in B^X$ , the probability measure  $\nu_{b,x}$  on  $X$  is supported on the fiber  $\{x' \in X : V_{b,x'} = V_{b,x}\}$ .

**Proposition 7.6.** *In the two cases of §6.1, for  $\beta^{\tau, X}$ -a.e.  $(b, x) \in B^X$ , the probability measure  $\nu_{b,x}$  is  $V_{b,x}$ -invariant and has the equivariance property  $\nu_{b,x} = b_{0*}\nu_{Tb, b_0^{-1}x}$ .*

*Proof.* The first assertion follows from Proposition 4.3.

The second assertion follows from the equality  $\nu_b = b_{0*}\nu_{Tb}$ , from Proposition 7.5, and from the disintegration of measures.  $\square$

## 8. APPLICATIONS

*In this chapter we conclude the proof of Theorems 1.1 and 1.3 and their corollaries.*

**8.1. Invariance of stationary measures.** *We keep the notations of §6.1 and we conclude this section with the classification of stationary measures on  $X$ .*

**Proposition 8.1.** *In the two cases of §6.1, the probability measure  $\nu$  is the Haar measure on  $X$ .*

In order to deduce this from Proposition 7.6, We will need the following lemma. Let  $\alpha \in \mathcal{P}(X)$ .

In the first case of §6.1, we denote by  $S_\alpha$  the connected component of the identity in the stabilizer of  $\alpha$  in  $G$ , with respect to the action by translations on  $X = G/\Lambda$ .

In the second case of §6.1, we denote by  $S_\alpha$  the connected component of the identity in the stabilizer of  $\alpha$  in  $\mathbb{R}^d$  with respect to the translation action on  $X = \mathbb{T}^d$ .

In both cases, we set

$$\mathcal{F} = \{\alpha \in \mathcal{P}(X) : S_\alpha \neq \{1\} \text{ and } \alpha \text{ is supported on one } S_\alpha\text{-orbit}\},$$

and endow this collection with the weak-\* topology.

We note that the group  $G$  acts naturally on  $\mathcal{F}$ . Denote by  $\nu_0$  the Haar measure on  $X$ . Then  $\nu_0$  is an element of  $\mathcal{F}$ .

**Lemma 8.2.** *In both cases of §6.1, the only  $\mu$ -stationary Borel probability measure  $\eta$  on  $\mathcal{F}$  is  $\delta_{\nu_0}$ .*

*Proof.* We can suppose that  $\eta$  is  $\mu$ -ergodic. We will distinguish the two cases:

**First case of §6.1.** In this case we have  $X = G/\Lambda$ .

By [15, Thm. 1.1], the set  $\mathcal{G}$  of  $G$ -orbits in  $\mathcal{F}$  is countable.

The image  $\bar{\eta}$  of  $\eta$  in  $\mathcal{G}$  is a  $\mu$ -stationary ergodic probability measure, on a countable set. By Lemma 8.3, the probability measure  $\bar{\eta}$  has finite support.

Since  $\eta$  is  $\mu$ -ergodic, it is supported on a unique orbit  $G\alpha \cong G/G_\alpha \subset \mathcal{F}$ . By definition of  $\mathcal{F}$ , the group  $G_\alpha$  is not discrete. Since  $G_\alpha$  contains a lattice, it is unimodular. By Proposition 6.7,  $G_\alpha = G$ . The probability measure  $\nu$  is thus equal to  $\delta_{\nu_0}$ .

**Second case of §6.1.** In this case we have  $X = \mathbb{T}^d$ .

We denote by  $\mathcal{G}$  the set of nontrivial tori in  $X$  and for  $Y \in \mathcal{G}$ , we denote by  $\mathcal{F}_Y$  the set of measures which are translates of the Haar probability measure on  $Y$ . The space  $\mathcal{F}$  is thus a countable union of compact subsets  $\mathcal{F}_Y$ .

The image  $\bar{\eta}$  of  $\eta$  in  $\mathcal{G}$  is a  $\mu$ -stationary ergodic probability measure on a countable set. By Lemma 8.3, it has finite support  $Y_1, \dots, Y_n$  and  $\Gamma$  permutes the subspaces  $V_1, \dots, V_n$  which are the tangent directions of the tori  $Y_1, \dots, Y_n$ .

Since the action of  $\Gamma$  is strongly irreducible, we necessarily have  $V = V_1 = \dots = V_n$ , which is what we had to prove.  $\square$

We will use the following classical result.

**Lemma 8.3.** *Let  $\Gamma$  be a group acting on a countable space  $X$  and let  $\mu$  be a probability measure on  $\Gamma$ . Any  $\mu$ -stationary and  $\mu$ -ergodic measure  $\nu$  is  $\Gamma$ -invariant and finitely supported.*

*Proof of Lemma 8.3.* Let  $Y$  be the set of points of  $X$  with maximal mass (w.r.t.  $\nu$ ). Then  $Y$  is finite. The equality  $\nu = \mu * \nu$  and the maximum principle imply that for  $\mu$ -a.e.  $\gamma \in \Gamma$ ,  $\gamma^{-1}Y \subset Y$  and hence  $\gamma^{-1}Y = Y$ . Since  $\nu(Y) > 0$  and  $\nu$  is  $\mu$ -ergodic,  $\nu(Y) = 1$ .  $\square$

*Proof of Proposition 8.1.* By Proposition 7.5, the fruit of our efforts, for  $\beta^X$ -a.e.  $(b, x) \in B^X$ , the subgroups  $V_{b,x}$  are nontrivial.

The principal interest in the set  $\mathcal{F}$  is that *it contains all of the probability measures invariant and ergodic under a connected nontrivial unipotent subgroup*. This results from Ratner's work [15] in the first case and is elementary in the second case.

For  $\beta^X$ -a.e.  $(b, x) \in B^X$ , the decomposition of  $\nu_{b,x}$  into  $V_{b,x}$ -ergodic components can thus be written simultaneously in the form

$$\nu_{b,x} = \int_X \zeta(b, x') d\nu_{b,x}(x'), \quad (8.1)$$

where  $\zeta : B^X \rightarrow \mathcal{F}$  is a  $\mathcal{B}^X$ -measurable map such that, for  $\beta^X$ -a.e.  $(b, x) \in B^X$ , the restriction of  $\zeta$  to the fiber  $\{(b, x') : V_{b,x'} = V_{b,x}\}$  is constant along the  $V_{b,x}$ -orbits.

The uniqueness of the ergodic decomposition, and Propositions 7.5 and 7.6, prove that, for  $\beta^X$ -a.e.  $(b, x) \in B^X$ , we have

$$\zeta(b, x) = (b_0)_* \zeta(T^X(b, x)). \quad (8.2)$$

By Lemma 3.2(e), the image probability measure  $\eta = \zeta_* \beta^X$  is therefore a  $\mu$ -stationary probability measure on  $\mathcal{F}$ . By Lemma 8.2, this probability measure is the Dirac mass on  $\nu_0$ . In other words,  $\zeta(b, x)$  is  $\beta^X$ -almost surely equal to  $\nu_0$ , so that  $\nu = \nu_0$ .  $\square$

*Proof of Theorems 1.1 and 1.3.* Recall that, in the second case, we have denoted by  $G$  the Zariski closure of  $\Gamma_\mu$  in  $\mathrm{SL}(d, \mathbb{R})$ . Lemma 8.5 below shows that  $G$  is also semi-simple.

In both cases, Lemma 8.4 below makes it possible to assume that  $G$  is a semi-simple noncompact Lie group. One can then apply Proposition 8.1 to conclude that  $\nu$  is  $G$ -invariant.  $\square$

We have used above the following two easy lemmas.

**Lemma 8.4.** *Let  $K$  be a metrizable compact group acting in Borel fashion on a Borel space  $X$ , and let  $\mu$  be a Borel probability measure on  $K$ . Then any  $\mu$ -stationary Borel probability measure  $\nu$  on  $X$  is invariant under the group  $\Gamma_\mu$  generated by the support of  $\mu$ .*

*Proof.* By Varadarajan's theorem [17, Prop. 2.1.19], we may suppose that  $X$  is compact and that the action is continuous. We may also suppose that  $\nu$  is  $\mu$ -ergodic. It is then supported on a unique  $K$ -orbit  $Kx_0$ . We can therefore consider  $\nu$  to be an  $H$ -invariant measure on  $K$ , for the action of  $H$  on the right, where  $H$  is the stabilizer of  $x_0$ . This lifted probability measure is also  $\mu$ -stationary. It remains to treat the case  $X = K$ .

Up to convolving  $\nu$  on the right by an approximate identity, we can suppose that  $\nu$  is absolutely continuous with respect to Haar measure, with a continuous density. We can thus think of  $\nu$  as an element of  $L^2(K)$  satisfying  $\mu * \nu = \nu$ . But in a Hilbert space, the average of vectors of a fixed norm has norm strictly smaller, unless the vectors being averages are equal to each other. This proves that  $\nu$  is  $\Gamma_\mu$ -invariant.  $\square$

**Lemma 8.5.** *Let  $\Gamma$  be a subsemigroup of  $\mathrm{SL}_d(\mathbb{Z})$  which acts strongly irreducibly on  $\mathbb{R}^d$ . Then its Zariski closure  $G$  in  $\mathrm{SL}(d, \mathbb{R})$  is a semisimple group.*

*Proof.* We can suppose that  $G$  is Zariski-connected. Since the representation of  $G$  on  $\mathbb{R}^d$  is irreducible,  $G$  is a reductive group. Since  $G$  is made of matrices of determinant 1, its center  $Z$  is compact. We need to show that  $Z$  is finite.

Suppose by contradiction that  $Z$  is infinite. The commutant of  $G$  in  $\mathrm{End}(\mathbb{Q}^d)$  is then an imaginary quadratic extension of  $K$  of  $\mathbb{Q}$ . We can then regard  $\mathbb{Q}^d$  as a  $K$ -vector space. The determinant map  $g \mapsto \det_K(g)$  embeds  $\Gamma$  in the group of units  $U_K$  of  $K$ . Since  $U_K$  is finite, the determinant map also embeds  $G$  in  $U_K$ . Therefore  $Z$  is finite, a contradiction.  $\square$



**8.2. Invariant measures.** *In order to deduce the corollaries of our theorems, we need to conveniently choose the measure  $\mu$ .*

*Proof of Corollaries 1.2(a) and 1.4(a).* Since  $G$  is simple, any Zariski dense subsemigroup  $\Gamma$  contains a finitely generated subsemigroup  $\Gamma'$  which is also Zariski dense. Denote by  $g_1, \dots, g_\ell$  a set of generators of  $\Gamma'$  and let  $\mu = \frac{1}{\ell}(\delta_{g_1} + \dots + \delta_{g_\ell}) \in \mathcal{P}(G)$ .

Let  $\nu$  be a non-atomic probability measure on  $X$  which is invariant under  $\Gamma$ . Then it is  $\mu$ -stationary. By Theorem 1.1 it is  $G$ -invariant, as required.  $\square$

**8.3. Closed invariant subsets.** *In order to prove corollaries 1.2(b) and 1.4(b), we will need the following lemma.*

**Lemma 8.6.** *In the two cases of §6.1, the collection of finite  $\Gamma$ -invariant subsets of  $X$  is countable.*

*Proof.* As before, we may suppose that  $\Gamma$  is finitely generated. Since  $\Gamma$  has countably many finite-index subgroups, it suffices to show that the points of  $X$  which are fixed by some subgroup  $\Delta$  of  $\Gamma$  are isolated. The last assertion follows from the fact that in any neighborhood of a fixed point, the linearization of the action of  $\Delta$  is its action on  $V$ , and since the action of  $\Gamma$  on  $V$  is strongly irreducible,  $\Delta$  does not have nonzero fixed vectors in  $V$ .  $\square$

*Proof of Corollaries 1.2(b) and 1.4(b).* We may again suppose that  $\Gamma$  is finitely generated. We then denote, just as in the proof of point (a), that  $\mu$  is the probability measure given by  $\mu = \frac{1}{\ell}(\delta_{g_1} + \dots + \delta_{g_\ell})$ , where  $g_1, \dots, g_\ell$  are a set of generators of  $\Gamma$ . Let  $F$  be an infinite closed  $\Gamma$ -invariant subset of  $X$ . By Lemma 8.6, we can construct an increasing sequence  $F_1 \subset F_2 \subset \dots \subset F_i \subset \dots$  of finite  $\Gamma$ -invariant subsets (possibly empty) of  $X$ , such that every finite  $\Gamma$ -invariant subset is contained in one of the  $F_i$ . Since  $F$  is infinite, we can choose pairwise distinct points  $x_1, x_2, \dots$  of  $F$  such that  $x_i$  is not in  $F_i$  for each  $i$ .

By Proposition 6.4, regarding recurrence off of finite subsets, there is a collection  $(K_i)_{i \geq 0}$  of compact subsets such that for each  $i$ ,  $K_i$  is contained in  $F_i^c$  and such that for all  $j \geq 1$ , there is an integer  $M_j$  such that for  $n \geq M_j$  and  $i \leq j$ ,

$$(\mu^{*n} * \delta_{x_j})(K_i^c) \leq \frac{1}{i}. \quad (8.3)$$

Setting  $n_j = jM_j$ , we introduce the Birkhoff-Kakutani averages

$$\nu_j = \frac{1}{n_j}(\mu * \delta_{x_j} + \dots + \mu^{*n_j} * \delta_{x_j}). \quad (8.4)$$

We have, for all  $i \leq j$ ,

$$\nu_j(K_i^c) \leq \frac{M_j}{n_j} + \frac{n_j - M_j}{n_j} \frac{1}{i} \leq \frac{2}{i}. \quad (8.5)$$

Condition (8.5) ensures that any accumulation point of the sequence  $(\nu_j)$  for weak-\* convergence of Borel probability measures, is a probability measure which gives no mass to the subsets  $F_i, i \geq 1$ . If  $\nu_\infty$  is such an accumulation point,  $\nu_\infty$  is then a  $\mu$ -stationary Borel probability measure satisfying  $\nu_\infty(F) = 1$  and  $\nu_\infty$  is non-atomic, by Lemma 8.3. According to Theorems 1.1 and 1.3,  $\nu_\infty$  is Haar measure. This implies the required equality  $F = X$ .  $\square$

**8.4. Equidistribution of finite orbits.** *The same arguments lead to a proof of equidistribution of finite orbits.*

*Proof of Corollaries 1.2(c) and 1.4(c).* We may again suppose that  $\Gamma$  is generated by the finite support of the measure  $\mu$ . We will show that the sequence of  $\Gamma$ -invariant measures

$$\nu_j = \frac{1}{\#X_j} \sum_{x \in X_j} \delta_x$$

converges weak-\* to the Haar probability measure on  $X$ . By point (a), we just have to show that any weak limit  $\nu_\infty$  of the sequence  $(\nu_j)$  is a probability measure which gives zero mass to finite orbits. The proof relies on the phenomenon of recurrence off of finite orbits. This is analogous to point (b) and we keep the notations  $F_i$  and  $K_i$ .

Since the finite  $\Gamma$ -orbits  $X_j$  are distinct, we can suppose after passing to a subsequence that for every  $j \geq i$ , we have  $\nu_j(F_i) = 0$ . Since  $\nu_j$  is  $\Gamma$ -invariant, for any  $n \geq 0$ , we have  $\mu^{*n} * \nu_j = \nu_j$  and therefore, as in (b), for any  $j \geq i$ ,  $\nu_j(K_0^c) \leq \frac{1}{i}$ . We deduce that for all  $i \geq 0$ , we have  $\nu_\infty(K_i^c) \leq \frac{1}{i}$ . This implies that firstly,  $\nu_\infty$  is a probability measure, and secondly, that  $\nu_\infty(F_i) = 0$  for all  $i$ , and therefore that  $\nu_\infty$  is Haar measure.  $\square$

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